Asset Prices, Global Portfolios, and the International Financial System

Maxime Sauzet*

November 23, 2020
Link to latest version

I characterize the global solution to the international portfolio problem in full generality, a long-standing open issue in international finance. In this two-country two-good environment, investors have recursive preferences and a bias in consumption towards their local good. The framework highlights the role of the allocation of wealth across international investors for portfolios, asset prices, and risk sharing, an aspect that had received little emphasis in such a setting. The influence of the allocation of wealth grows especially as markets become imperfectly integrated, and as investor heterogeneity rises – be it through a larger home bias in consumption, the introduction of labor income, or asymmetries in preferences – to the point where it can match or surpass the impact of fundamentals. The framework lends itself to several applications and extensions. In particular, I show that it can replicate a number of facts about the structure and dynamics of the international financial system, and of asset returns in that context.

Keywords: International Portfolio Choice, Asset Pricing, International Finance and Macroeconomics, International Financial System, Wealth Allocation.

JEL codes: E0, F3, F4, G1.

*University of California, Berkeley. Email: msauzet@berkeley.edu. For invaluable support and guidance, I thank my advisor Pierre-Olivier Gourinchas, as well as Nicolae Gârleanu, Martin Lettau, Hélène Rey, and Jón Steinsson. For valuable comments and discussions, I thank Markus Brunnermeier, Xiang Fang, Chen Lian, Matteo Maggiori, Ulrike Malmendier, Sebastian Merkel, Thomas Mertens, Emi Nakamura, Maurice Obstfeld, David Sraer, Richard Stanton, Annette Vissing-Jørgensen, Johan Walden, and seminar participants at the Department of Economics, and the Haas School of Business of the University of California, Berkeley. All remaining errors are my own.

1. Introduction

> *"At the moment, we have no integrative general-equilibrium (monetary) model of international portfolio choice, although we need one."*
>
> Obstfeld (2004)

More than a decade later, the international portfolio choice problem remains a long-standing open issue in international finance to which the literature only provides a piecemeal answer. My first contribution in this paper is to characterize the global solution to the international portfolio choice problem in full generality. The framework is a well-suited building block towards several applications and extensions. My second contribution focuses on one of them and shows that the model can be used to capture a number of stylized facts about the structure and dynamics of the international financial system, and of asset returns in that context.

The main economic message from the first contribution is that the allocation of wealth across investors matters in a general international portfolio choice setting. This finding resonates with an emerging theme in the broader economic literature that has recently emphasized the role of the wealth distribution in determining economic outcomes in macroeconomics (e.g. Brunnermeier and Sannikov, 2014, Kaplan et al., 2018), finance (e.g. Gomez, 2017, Lettau et al., 2019, Greenwald et al., 2020), and economics more generally (e.g. Piketty and Zucman, 2014). In other words, "capital is back" in this setting too: the allocation of wealth across international investors has a prime role in driving asset prices, portfolios, and risk sharing, an aspect that had received little emphasis thus far.

To derive this result, I adapt recent advances in multi-agent continuous-time asset pricing models to a two-country, two-good economy in which investors have recursive preferences and a bias in consumption towards their local good. This allows me to overcome two main limitations in the international portfolio choice literature.

First, while a majority of contributions rely on special cases to facilitate the resolution, I allow for general recursive preferences and an arbitrary degree of substitutability across goods. The former matters because (i) recursive preferences are not log so that investors are not myopic and their portfolios feature hedging demands that have a prime role in this international context, and (ii) recursive preferences are not constant relative risk aversion (CRRA), which leads the allocation of wealth across investors to become a state variable in its own right that has an important impact beyond current fundamentals.[1] An arbitrary degree of substitutability across goods ensures, by moving away from the case of unitary elasticity of substitution, that asset returns are not perfectly correlated so that the portfolio

[1] Specifically, recursive preferences break the link between the elasticity of intertemporal substitution and the inverse of risk aversion. They also help in generating quantitatively more plausible risk premia while maintaining a reasonable risk-free rate. Hedging terms are absent more generally as long as the risk aversion is equal to one.

choice between them is well-defined.[2] Throughout, the generality of the specification allows to study the impact of a number of important dimensions of preferences.

Second, while most contributions have relied on low-order local approximations, I solve the model using a global solution method. This makes it possible to fully trace out the evolution of economic variables with the state of the economy, in sharp contrast to local methods that mostly capture evolutions in a small neighborhood of a specific state.[3] This innovation is particularly valuable in situations such as here in which economic outcomes turn out to be strongly state-dependent, and in which policy functions can be very non-linear as a result of heterogeneity, or imperfect risk sharing. In addition, because increasing the order of approximation is notoriously cumbersome for the type of local methods that have been used in the literature, most contributions have focused on so-called zero-order (i.e. steady-state) portfolios. Such portfolios, which are constant, are silent on any time variation in investors' positions. Instead, the global method in this paper naturally captures their dynamics, an aspect that is not innocuous: like other outcomes, portfolios are inherently time-varying. For instance, the bias in portfolio holdings towards the home or foreign asset that emerges in equilibrium is strongly reinforced as the wealth share of an investor decreases, and the relative portfolio weights of different assets also vary substantially with the relative supply of goods in the world economy.

More generally, I augment the framework in a number of ways, e.g. by introducing labor income as a constant share of output, imperfect financial integration, or asymmetries in preferences, which allow me to analyze the international portfolio choice problem in a variety of contexts. This is made possible in part by the fact that throughout, I solve for the *decentralized* equilibrium to the economy so that I am able to study cases in which the standard planner solution (that have been popular in the literature) cannot be used.

The allocation of wealth impacts the economy in two ways.

Its first role is that of a state variable in its own right, beyond current fundamentals, which captures the average international investor. The profile of this average investor varies significantly depending on which international investor owns a larger share of world wealth, so that the allocation of wealth directly impacts asset prices, portfolios, and other economic outcomes. Specifically, because the domestic investor has a preference towards the local good, an increase in her wealth share puts upward pressure on the price of the domestic

[2]The case of unitary elasticity of substitution across goods has received considerable attention in the literature since the seminal contribution of Cole and Obstfeld (1991). For instance, it is assumed in Pavlova and Rigobon (2007, 2008, 2010), Colacito and Croce (2011, 2013), Maggiori (2017), and Colacito et al. (2018), among others.

[3]Under a set of assumptions, local methods could be used to study an economy further in the state space, cf. for instance Mertens and Judd (2018). However, such methods remain difficult to use in an international portfolio context due to the portfolio indeterminacy that arises in the corresponding deterministic economy. More generally, defining the state around which to approximate the equilibrium is also non-trivial. The literature has focused on using the symmetric economy as an approximation point, but this might not be a well-defined steady state in an international context in particular in the presence of imperfect risk sharing, incomplete markets, and non-stationarity. The global method in this paper circumvents all those difficulties naturally.

good, so that the returns on the domestic asset, which pays in that good, increase.[4] In turn, the effect on risk premia is reflected on portfolios: as the wealth share of an investor increases, the bias in their equity holdings towards the domestic or foreign asset that obtains in equilibrium diminishes. Those effects are large, with investors strongly tilting their portfolios when their wealth share is small, but converging towards holding the market portfolio when they dominate world wealth. This stands in sharp contrast to the portfolios that have been the main focus of the literature, which are constant and computed solely at the point in which both investors own equal wealth.

Where does the bias in portfolio holdings come from? Because preferences are not log, investors tilt their portfolios to hedge against risks in the economy.[5] To start with fundamentals, the hedging of shocks to relative supply leads both investors to prefer assets whose returns are large when their preferred good is rare, given that their marginal value of wealth is high in such circumstances. When goods are good enough substitutes, the foreign asset pays more when the relative supply of the domestic good is small, because this means that the relative supply of the *foreign* good is large. This leads to a foreign bias in equity holdings for both investors as the foreign asset returns are large when their respective marginal value of wealth is high. On the other hand, when goods are poor substitutes, the impact on goods prices of consumer demand is such that an asset pays more when the relative supply of the *other* good is large. This therefore results in a home bias in equity holdings. Due to the fact that relative prices such as the exchange rate are strongly related to relative supply, those findings are consistent in this more general framework with the hedging of real exchange rate risk that has been the focus in the literature. Importantly, because standard estimates of the elasticity of substitution between goods puts us in the former case, turning the counterfactual foreign bias that obtains into a home bias in equity holdings like in the data, will rely on the introduction of another plausible channel, imperfect financial integration, that I discuss below.[6]

What about the allocation of wealth? Because it impacts relative prices and asset returns, wealth share risk is also hedged by international investors. Under perfect risk sharing, this turns out to reinforce the bias in portfolio holdings towards the domestic or foreign asset discussed above. This owns to (i) the negative relationship that obtains in equilibrium between wealth share and relative supply, and (ii) the fact that the relative marginal value of wealth of an investor tends to increase with their wealth share.[7] (i) emerges regardless of

[4]This is so as long as goods are good enough substitutes, as discussed below.

[5]Hedging terms are absent more generally as long as the risk aversion is equal to one.

[6]The hedging of real exchange risk has a long history in the international portfolio choice literature. Cf. Coeurdacier (2009) for a recent take, and Obstfeld (2007) and Coeurdacier and Rey (2013) for surveys. I discuss it in more detail in Section 3.4. Coeurdacier and Rey (2013) also summarize recent empirical findings on the home bias in equity holdings. The impact of the elasticity of substitution is discussed at length throughout Section 3 but modern estimations such as those in Imbs and Méjean (2015) and in the international trade literature put it firmly in the case of goods being good substitutes.

[7](ii) comes from the fact that as the wealth share of an investor increases, the impact of that investor on relative prices grows and the price of its preferred good increases, which makes them relatively worse-off. Intuitively, this is also consistent with this investor growing more dominant in world wealth so that diversifying risks with the other, increasingly small, investor is more difficult.

good substitutability because an investor allocates more wealth to the asset – domestic or foreign – whose returns are large when the physical supply of their preferred good is low. As a result, a shock that tends to improve the relative supply of their preferred good necessarily leads their preferred asset to do poorly, so that their share of wealth decreases. But because in consequence their marginal value of wealth also decreases in the wealth share dimension, an investor values the asset that pays in those conditions even less and therefore overweights its already preferred asset further. In short: the hedging of wealth share risk reinforces the bias in portfolio holdings under perfect risk sharing.

Quantitatively, the impact of the allocation of wealth remains modest in a symmetric baseline under perfect risk sharing with the wealth share evolving in a narrow band around a broad direction given by fundamentals. However, this impact grows tremendously as soon as markets become imperfectly integrated, and as investors become more heterogeneous. In both cases, the role of the allocation of wealth for asset prices, portfolios, and other economic outcomes, can be on par with or surpass that of fundamentals captured by the relative supply.

Introducing imperfectly integrated markets in this economy is particularly relevant because investing internationally comes with a number of frictions – be they legal, technical, informational, or otherwise. I capture those frictions in a parsimonious way as a tax on foreign dividends, generalizing Bhamra et al. (2014).[8] The formulation allows me to study the effect of a range of financial integration degrees without having to take a specific stance on the source of the underlying imperfections.

By making the foreign asset less attractive, due to the direct required payment of the tax as well as a modest general equilibrium effect, imperfect financial integration can rapidly overcome the foreign bias in equity holdings that obtains in the baseline and deliver a home bias in equity holdings in line with empirical observations. When this happens, the impact of the allocation of wealth is also strongly reinforced, consistent with the fact that risk sharing becomes imperfect so that insuring against risks in the economy becomes more difficult for investors.[9] The allocation of wealth has a larger direct effect as a state variable, but the impact is also visible in terms of hedging demands: insuring against shocks to their wealth share becomes as important a driver of investors' portfolios as the hedging of fundamentals. In addition, the hedging of wealth share risk now contributes to obtaining a *home* bias in equity holdings, in contrast to the baseline in which it reinforced the foreign bias coming from fundamentals. This occurs because of the overpowering effect of imperfect financial integration on risk premia, which makes the local asset more attractive and therefore on

[8]Cf. also the seminal contribution of Basak and Gallmeyer (2003), who study a dynamic asset pricing model with asymmetric dividend taxation and a unique risky asset in a one-country one-good setting. As shown in Gârleanu et al. (2020), models with investment taxes constitute an equivalent, but substantially simpler, way to capture a rich set of impediments to financial trade. As such, the tax is meant to capture not only actual differential tax treatments or transaction costs for investment across countries, but more generally any friction that prevent investors from freely participating in foreign markets.

[9]Imperfect risk sharing arises because the tax makes the opportunity sets of the two investors different so that their stochastic discount factors are no longer perfectly correlated. Another consequence is that the standard planner solution that has been popular in the literature can no longer be used.

average more prevalent in the local portfolio. This in turn yields a switch in the equilibrium relationship between wealth share and relative supply: a shock that increases local relative supply also leads the local asset to do well, and therefore the local wealth share to increase. As a result, the hedging of wealth risk flips sign and contributes positively to obtaining a home bias in equity holdings.[10]

Importantly, the calibration of preferences has a substantial effect on the ultimate potency of imperfect financial integration, with the elasticity of intertemporal substitution taking center stage. When this elasticity is low, modest taxes on the order of $\tau = 7$ to 10% are enough to deliver a home bias in equity holdings like in the data, qualitatively throughout the state space, and quantitatively in at least some regions of it.[11] When the elasticity is high however, and even though the same mechanisms are at play, the effects are much more muted and for reasonable taxes, a foreign bias in equity holdings remains.[12] This additional novel result arises because the dividend yields on the two equity assets, which are the ultimate driver of the effect of the tax on returns, are significantly smaller in magnitude in this case. Economically, it reflects the fact that the elasticity of intertemporal substitution has a large impact on the extent of trading in the risk-free bond, which was unused under perfect risk sharing but becomes important with imperfect integration, as well as on the diversification benefits provided by the two equity assets that vary a lot both with parameters and with the state of the economy.

Taken together, those results confirm but qualify the findings in Bhamra et al. (2014) in this general setting with non-log preferences and home bias in consumption: imperfectly integrated markets *can* deliver portfolios consistent with the data *provided that* the elasticity of intertemporal substitution is moderate. From the perspective of the main application below, a realistic home bias in equity holdings will therefore be generated by combining a moderate elasticity of intertemporal substitution with modest taxes on foreign dividends.

The heterogeneity of investors is another factor that has a sizable effect on the equilibrium. This is visible even in a symmetric baseline calibration: as the degree of home bias in consumption increases, the fundamental level of heterogeneity between investors also increases. As a result, the quantitative impact of the allocation of wealth across those – now more different – investors grows. For instance, the hedging of wealth share risk becomes once again on par with that of fundamentals. The same observation is true when introducing labor income, which can strongly reinforces the bias in portfolio holdings.[13] Heterogeneity

[10]This switch also has long-term consequences in terms of which investor survives in the long run. In addition, the dispersion of the wealth share in equilibrium increases with imperfect risk sharing so that the quantitative effect of the wealth share is larger.

[11]E.g. the home bias can be made consistent with empirical measures reported in Coeurdacier and Rey (2013) around the symmetric point in the state space. Importantly, portfolios remain inherently state-dependent.

[12]Generating a home bias consistent with empirical observations requires implausible taxes as high as $\tau =$ 75% or 90%.

[13]In the spirit of Baxter and Jermann (1997), labor income tends to lead to a foreign bias in equity holdings in this setting because it is modeled as a constant share of the output of each tree. More general specification such as a time-varying share in the spirit of Coeurdacier and Gourinchas (2016) or idiosyncratic labor income risk as in Kaplan et al. (2018) are interesting avenues for further exploration.

in the form of asymmetric preferences is also especially potent, in particular in terms of its effect on risk premia, and is explored in detail in the main application below.

To summarize this first contribution, the impact of the wealth share grows markedly with the degree of imperfect financial integration, and the degree of investor heterogeneity, to the point where it can come to be on par with or surpass the effect of fundamentals on portfolios, asset prices, and other economic outcomes. This reiterates the main message: capital is back in this international economy too![14] On a more theoretical note, the results emphasize both the strong state-dependence of most economic variables in this environment, and the vital impact of the calibration of preferences. This makes the novel framework presented in this paper, which is based on a global solution method and allows for general recursive preferences including asymmetries, particularly suited to study this economy.

Because of its generality, the framework in this paper represents a versatile building block towards several applications and extensions. My second contribution focuses on one main application, in which I show that the model can reproduce a number of stylized facts about the structure and dynamics of the international financial system, and in particular the role of the United States, and of asset returns in this context.[15]

The domestic country is now taken to represent the United States, the country at the center of the international financial system, and its representative investor is assumed to display a higher tolerance for risk. This assumption, in the spirit of Caballero et al. (2008), Gourinchas et al. (2017), and Maggiori (2017), is meant to capture the greater development and depth of U.S. financial markets in the general context of this paper. Like in Gourinchas et al. (2017), and Maggiori (2017), by making the country as a whole better able and willing to carry financial risk in the world economy, this asymmetry naturally replicates its average external position (Fact 1, Gourinchas and Rey, 2007b): the United States plays the role of the world banker, by borrowing in safe securities from the rest of the world, and investing in risky assets internationally. This large negative net foreign asset position is associated with higher excess returns on the external balance sheet of the country on average, given the higher share of risky assets that pay more in expectation: this is the exorbitant privilege of the world banker (Fact 2, Gourinchas et al., 2017). Importantly, the economy also still features two meaningfully different equity assets and a modest degree of imperfect financial integration delivers a home bias in equity holdings broadly consistent with empirical observations (Fact 4, Coeurdacier and Rey, 2013).

The framework does not only replicate facts about external portfolios on average however, and the asymmetry in risk tolerance yields a number of predictions about the *dynamics* of the international financial system that are strongly borne out in the data. As a crisis hits, the center country is impacted particularly severely due to its high allocation to risky assets, so

[14]These results are also reminiscent of recent findings in the price impact literature, in which quantities, represented here by the portfolios held by each investor and captured in aggregate by the wealth share, strongly impact asset prices and risk compensations. Contributions in this spirit include Kouri (1982), Jeanne and Rose (2002), Hau and Rey (2006), and more recently Gabaix and Maggiori (2015), Camanho et al. (2018), Gabaix and Koijen (2020), and Koijen and Yogo (2020).

[15]For an overview and additional references, those facts are summarized in Section 4.1.

that it transfers a large amount of wealth to the rest of the world. This exorbitant duty is the flip side of its exorbitant privilege in normal times: the United States must become the world insurer in times of trouble (Fact 3, Gourinchas et al., 2017). In addition, by worsening the wealth position of the risk-tolerant world banker, the shock leads to a sharp increase in global risk aversion, which in turn pushes up all risk premia and Sharpe ratios worldwide. These two markers are reminiscent of some aspects of the Global Financial Cycle (Fact 5, Rey, 2013, Miranda-Agrippino and Rey, 2020), for which a general equilibrium exploration had remained elusive. Those patterns are representative of the type of global risk-off scenarios that typically occur in times of global crisis such as most recently in the Great Recession of 2008 or the Global Pandemic of 2020.

In addition, the model allows to study the evolution of portfolios as a response to those shocks, and can shed light on the process of external adjustment of the center country. For the latter, while its net foreign asset position strongly deteriorates following the shock, the sharp increase in risk premia that occurs simultaneously highlights the role of valuation effects as proposed in Gourinchas and Rey (2007a) in this situation: the higher expected returns on its global portfolio ease some of the pressure on the domestic country to balance its external position in the short term. This negative relationship between net foreign asset position and expected risk premia therefore replicates the type of predictability relationship between the two documented in Gourinchas and Rey (2007a), and extended to more recent data in Gourinchas et al. (2019) (Fact 6).[16]

From an asset pricing perspective, the model speaks to a number of facts about asset returns dynamics in this international environment. Namely, risk premia, Sharpe ratios – and to some extent volatilities and correlations in a relevant region of the state space – are all countercyclical in the sense that they increase following the shock, consistent with a wide range of evidence notably for the United States (Fact 7, Lettau and Ludvigson, 2010, among others). Those patterns are the reflection of the type of dynamics emerging in asset pricing settings with heterogeneous agents (e.g. Weinbaum, 2009), in an economy in which there are also two goods, two assets, and a home bias in consumption. Importantly, those patterns are driven for a large part not by changes in the quantity of risk but by the evolution of the compensation for risk, captured here by the time-varying global risk aversion. This is in line with a large literature that has seen changes in the price of risk emerge as a crucial explanation behind asset return predictability more generally.

Another value of studying those questions in the general framework of this paper is that it allows to perform a number of counterfactual exercises. For instance, I show that a mild decrease in the frictions in international markets can generate the secular decrease in home bias that has been documented in recent decades (Fact 4, Coeurdacier and Rey, 2013), as well as some of the increase in the financial synchronization that has been observed throughout the world over a long-time horizon but particularly in the last three decades (Fact 8, Jordà et al., 2019). A re-interpretation of the model at a lower frequency could

[16] In the long run, the higher share of risky assets in the domestic portfolio also leads the domestic country to grow in world wealth and its net foreign asset position to become positive. This further alleviates the burden on the necessity of short-term adjustment in times of crisis, and allows the country to run a more negative net foreign asset position for a while.

also be used to make sense of the secular decline in interest rate that has been observed worldwide, provided that the wealth share of the domestic risk-tolerant country decreases in the long run (Fact 9, Caballero et al., 2008, Hall, 2016). Finally, changes in the tax on foreign dividends, potentially asymmetric, could also be used to study the impact on global asset prices, portfolios, and risk sharing, of macroprudential policies aimed at curbing sudden international capital flows.

In summary of this second contribution, a seemingly small change in the specification of the model – the introduction of asymmetries in risk tolerance – generates a vast number of facts about the structure and dynamics of the international financial system and of asset returns, which are strongly borne out in the data. The model is also a well-suited building block for many potential extensions. The most promising among them are related to the introduction in an international setting of financial intermediaries of the type that has been discussed in the recent intermediary asset pricing literature e.g. in Daníelsson et al. (2012), He and Krishnamurthy (2013), Adrian and Shin (2014), or Adrian and Boyarchenko (2015). Illustrations are briefly discussed in Section 4.5 and Appendix E, for instance with the inclusion of a global asset manager (Sauzet, 2020a). From the perspective of extensions, solving for the decentralized equilibrium of this economy like I do in this paper will prove particularly valuable: the framework is readily set to tackle a wide range of market structures beyond imperfect risk sharing. In addition, the implementation of those extensions will likely require higher-dimensional methods such as the "projection methods via neural networks" being developed in Sauzet (2020b). I leave all these promising avenues for future research.

Related literature

This paper contributes to two main strands of literature.

First, I contribute to the literature on multi-agent asset pricing models, which has a long and distinguished history since the seminal contributions of Dumas (1989, 1992), Wang (1996), Basak and Cuoco (1998), Chan and Kogan (2002), and more recently Brunnermeier and Pedersen (2009), Weinbaum (2009), Brunnermeier and Sannikov (2014), Gârleanu and Pedersen (2011), Chabakauri (2013), Gârleanu and Panageas (2015), Drechsler et al. (2018). This literature is also related to the modern literature on heterogeneous agents in closed-economy macroeconomics such as Kaplan et al. (2018). To those contributions, I bring two goods, two assets, two countries, as well as a home bias in consumption. The home bias in consumption is particularly important because it introduces a fundamental level of heterogeneity between investors even absent asymmetries, and is responsible for most mechanisms in the economy including the rise of a substantial bias in portfolio holdings through hedging demands, the shape and comovement of risk premia, and a well-defined exchange rate. As such, this is one of the main differences with the international model of Brunnermeier and Sannikov (2015, 2019). Having two assets also relates my paper to contributions with multiple securities but one agent e.g. Cochrane et al. (2008), Martin (2013).

Most related to my contribution are those of Pavlova and Rigobon (2007, 2008, 2010) and Stathopoulos (2017), who study a pure exchange economy similar to mine, but in which preferences are log and the elasticity of intertemporal substitution across goods is equal to one. The combination of those assumptions leads the allocation of wealth to be constant, equity assets to be perfectly correlated in the absence of demand shocks, and hedging demands to be absent due to myopic portfolios. All three are important dimensions that arise in my framework once I allow for general recursive preferences and an arbitrary elasticity of substitution between goods. I therefore see my contribution has the natural continuation of this earlier research effort.

Breaking those limitations does not come without a cost however, and solving the model requires a whole new set of methods compared to those papers. In particular, the resolution of my framework is based on global projection methods, as presented in Judd (1992, 1998), the NBER Summer SI Lecture by Fernández-Villaverde and Christiano (2011), or Parra-Alvarez (2018), and as applied to multi-agent models for instance in Drechsler et al. (2018), Fang (2019), or Kargar (2019). The approximation is based on Chebyshev polynomials and orthogonal collocation, although in concurrent work, I am also developing a natural extension based on neural networks (Sauzet, 2020b, cf. Section 4.5).[17]

In addition, I also introduce asymmetries in preferences, labor income in the form of a constant share of output as in Baxter and Jermann (1997), and most importantly, imperfect financial integration. The latter is captured in a parsimonious way as a tax on foreign dividends by generalizing Bhamra et al. (2014) to a non-log environment that also features home bias, and following the seminal contribution of Basak and Gallmeyer (2003) who study a dynamic asset pricing model with asymmetric dividend taxation and a unique risky asset in a one-country one-good setting. Compared to Bhamra et al. (2014), the introduction of general preferences makes a significant difference: imperfect risk sharing has a large impact *provided that* the elasticity of intertemporal substitution is modest, a novel insight. In addition, I use a global solution instead of relying on local approximations, and am able to study the effect on the exchange rate and of hedging terms. Theoretically, the use of a tax to capture a wide range of frictions is related to the work of Gârleanu et al. (2020), who show that models with investment taxes constitute an equivalent, but substantially simpler, way to capture a rich set of impediments to financial trade.

Other related papers include Cass and Pavlova (2004), Brandt et al. (2006), Martin (2011), and Maggiori (2017) that I discuss below, as well as Fang (2019) who focuses on a small open economy in which the rest of the world is taken as exogenous and in which investors do not have symmetric home bias. On the theoretical front, my paper is also related to contributions introducing recursive preferences in continuous-time e.g. Duffie and Epstein (1992), and contributions focusing on the existence and uniqueness of equilibria in the presence of multiple agents, and possibly multiple goods and incomplete markets e.g. Polemarchakis (1988), Geanakoplos and Polemarchakis (1986), Geanakoplos and Mas-

[17] I solve for the decentralized economy throughout, but the method of Dumas et al. (2000), based on a planner, could also be used in cases in which risk sharing is perfect.

Colell (1989), Geanakoplos (1990),Duffie et al. (1994), **?**, Anderson and Raimondo (2008), Hugonnier et al. (2012), Ehling and Heyerdahl-Larsen (2015).

Second, I contribute to the literature on the international portfolio problem. Specifically, the advances presented above allow me to characterize the general and global solution to the international portfolio choice problem, a long-standing issue in this literature since the seminar contributions of Stulz (1983), Dumas (1989, 1992), Cole and Obstfeld (1991), Baxter and Jermann (1997), Baxter et al. (1998), Obstfeld and Rogoff (2001), Obstfeld (2004), among many others. Obstfeld (2007) and Coeurdacier and Rey (2013) provide surveys.

To a large part of the more recent literature on the topic, such as Corsetti et al. (2008), Tille and van Wincoop (2010), Coeurdacier (2009), Devereux and Sutherland (2011), Evans and Hnatkovska (2012), Coeurdacier and Rey (2013), Coeurdacier and Gourinchas (2016), I bring (i) a solution that is global and does not rely on approximations. This allows to complete the picture and trace out the evolution of economic outcomes as we move away from the point of approximation (typically the symmetric point), which proves important in this context where variables are strongly state-dependent and potentially non-linear. I also bring (ii) general preferences, which allow to move away from special cases and study all situations under a unified framework (cf. also the discussion above of Pavlova and Rigobon, 2007, 2008, 2010, Stathopoulos, 2017). Compared to the limited number of contributions that have relied on global methods in similar settings e.g. Kubler and Schmedders (2003) (one country), Stepanchuk and Tsyrennikov (2015) (one good) Rabitsch et al. (2015), Coeurdacier et al. (2020) (one good), I bring (iii) continuous-time methods, which make it possible to study portfolio drivers, in particular hedging demands, asset prices and their conditional first and second moments, as well as the determinants of wealth and state variable dynamics, in ways that are inaccessible in a discrete-time formulation and therefore make continuous-time the natural tool of choice to study this type of questions. Finally (iv), to all, in addition to labor income as in Baxter and Jermann (1997) and asymmetries in preferences, I bring imperfect financial integration, which is an important topic in international finance but had not been studied thus far in a general international portfolio choice context.[18]

My contribution is also related to those of Colacito and Croce (2011, 2013), and Colacito et al. (2018), who introduce recursive preferences in an international context. Compared to those, output does not feature long-run risk dynamics. Instead, I bring in an arbitrary elasticity of substitution across goods, which makes the two equity assets no longer perfectly correlated so that the portfolio choice is no longer indeterminate in my context. More generally, I bring (i), (ii), (iii) and (iv) above to that economy.

Finally, the main application in this paper is in the spirit of Gourinchas and Rey (2007a,b), Caballero et al. (2008), Gourinchas et al. (2017), and Maggiori (2017) that I bring to the general international portfolio choice context of my framework. Papers related to the facts

[18]More general specification of labor such as a time-varying share in the spirit of Coeurdacier and Gourinchas (2016) or idiosyncratic labor income risk as in Kaplan et al. (2018) are interesting avenues for further exploration.

that this specialization of the model replicates were discussed previously, and are also mentioned in Section 4.1 with a summary of the stylized facts. Compared to Maggiori (2017) in particular, one main difference is the presence of recursive preferences compared to log, which ensures that the wealth share across international investors is not constant even in the absence of bankers, and that portfolios feature hedging demands that are crucial to study biases in equity holdings. In addition, the elasticity of intertemporal substitution is not equal to one, so that the returns on the two equity assets are not perfectly correlated and the portfolio choice is not indeterminate. I also introduce (iv) labor income, asymmetries, and imperfect risk sharing.

The paper is organized as follows. Section 2 describes the set-up of the economy, and introduces the two state variables that drive economic mechanisms: the wealth share of the domestic investor, and the relative supply of the two goods, i.e. fundamentals. Section 3 characterizes the solution to the model both theoretically, and by presenting the resulting equilibrium variables. It discusses in particular the role of the wealth share and how it grows as markets become less perfectly integrated, and agents become more heterogeneous. Section 4 presents the main application of the framework to modeling the international financial system, as well as possible extensions. Section 5 concludes. Additional material is provided in Appendix.

2. The Economy

This section presents the theoretical set-up. I introduce a pure-exchange economy with two countries, domestic and foreign (*), and two goods. Each country is populated by a representative investor with recursive preferences and whose consumption is biased towards the local good. I show that the equilibrium can be characterized as a function of two state variables: the wealth share of the domestic investor, x_t, and the relative supply of the two goods, y_t. The former captures the allocation of wealth between the two countries, and therefore the identity of the average investor in the world economy, while the latter captures fundamentals. The set-up is summarized in Figure F.1 in Appendix. Appendix A gathers additional results that are omitted in the main text.

Time is continuous and the horizon is infinite, $t \in [0, \infty)$. Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting a two-dimensional Brownian motion $\vec{Z} \equiv (Z, Z^*)^T \in \mathbb{R}^2$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ is the usual augmentation of the filtration generated by the Brownian motions, and $\mathcal{F} \equiv \mathcal{F}_\infty$.

2.1. Endowments, prices, assets

Each country hosts a tree, à la Lucas (1978), which produces a differentiated good with its own price. The output of each tree follows a geometric Brownian motion

$$\begin{aligned}\frac{dY_t}{Y_t} &= \mu_Y dt + \sigma_Y^T d\vec{Z}_t \\ \frac{dY_t^*}{Y_t^*} &= \mu_Y^* dt + \sigma_Y^{*T} d\vec{Z}_t\end{aligned} \tag{1}$$

The price of the domestic and foreign goods are p_t, p_t^*. The terms of trade is $q_t \equiv p_t^*/p_t$, defined as is standard so that an increase in q_t corresponds to a worsening of the terms of trade. The real exchange rate is $\mathcal{E}_t \equiv P_t^*/P_t$, defined as is standard so that an increase in $\mathcal{E}_t$ corresponds to a depreciation. P_t, P_t^* are the prices of the domestic and foreign consumption baskets discussed below. All prices are defined with respect to a global numéraire taken to be a CES-basket with weight a on the local good.[19]

Both trees are traded as equity assets, with returns given by

$$\begin{aligned}dR_t &= \frac{dQ_t}{Q_t} + \frac{p_t Y_t}{Q_t} dt = \frac{d\left(p_t Y_t/F_t\right)}{p_t Y_t/F_t} + F_t dt \equiv \mu_{R,t} dt + \sigma_{R,t}^T d\vec{Z}_t \\ dR_t^* &= \frac{dQ_t^*}{Q_t^*} + \frac{p_t^* Y_t^*}{Q_t^*} dt = \frac{d\left(p_t^* Y_t^*/F_t^*\right)}{p_t^* Y_t^*/F_t^*} + F_t^* dt \equiv \mu_{R*,t} dt + \sigma_{R*,t}^T d\vec{Z}_t\end{aligned} \tag{2}$$

where Q_t, Q_t^* are the equity prices, and $F_t \equiv p_t Y_t/Q_t$, $F_t^* \equiv p_t^* Y_t^*/Q_t^*$ are the dividend yields, for the domestic and foreign assets. Drifts $\mu_{R,t}, \mu_{R*,t}$, which measure conditional expected returns, and diffusion terms $\sigma_{R,t}, \sigma_{R*,t}$, which measure the loadings on the shocks and therefore the conditional volatilities, are obtained from Itô's Lemma and given in Appendix A.2.

The supply of each equity asset is normalized to unity, and there also exists an international bond in net zero supply, which is locally riskless in units of the numéraire. Its price is B_t, and the corresponding instantaneous interest rate is r_t, so that $dB_t/B_t = r_t dt$.

2.2. Preferences

The representative investor of each country has recursive preferences over consumption à la Duffie and Epstein (1992). This is in contrast to a large part of the literature that focuses on log or constant relative risk aversion (CRRA) utility. The former has the clear drawback that investors are myopic so that state variables are not hedged, and have therefore a limited impact on portfolios, asset prices, and other quantities of interest.[20] Contrary to the

[19] Specifically, I normalize $\left[a p_t^{1-\theta} + (1-a) p_t^{*1-\theta}\right]^{1/(1-\theta)}$ to unity.

[20] Hedging terms are absent more generally as long as the risk aversion is equal to one.

CRRA case, recursive preferences also allow to disentangle the risk aversion and elasticity of substitution of each investor, which is important to get closer to matching empirical moments. For the domestic investor, preferences are given by

$$V_t = \max_{\{C_{h,u}, C_{f,u}, w_{h,u}, w_{f,u}\}_{u=t}^{\infty}} \mathbb{E}_t \left[\int_t^{\infty} f\left(C_u, V_u\right) du \right] \tag{3}$$

$$f(C,V) \equiv \left(\frac{1-\gamma}{1-1/\psi} \right) V \left[\left(\frac{C}{\left[(1-\gamma)V\right]^{1/(1-\gamma)}} \right)^{1-1/\psi} - \rho \right]$$

where γ is the coefficient of relative risk aversion, $\psi \neq 1/\gamma$ the elasticity of intertemporal substitution, and ρ is the discount rate.

The consumption basket of each investor is composed of the two goods, which are combined according to an aggregator with constant elasticity of substitution θ, and is biased towards their own local good

$$C_t = \left[\alpha^{\frac{1}{\theta}} C_{h,t}^{\frac{\theta-1}{\theta}} + (1-\alpha)^{\frac{1}{\theta}} C_{f,t}^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \tag{4}$$

Two characteristics of the consumption baskets are noteworthy.

First, the elasticity of intertemporal substitution θ is in general not equal to unity. Due to its specificity, the case with $\theta = 1$, in which C_t collapses to a Cobb-Douglas aggregator, has received considerable attention in the literature since the seminal contribution of Cole and Obstfeld (1991).[21] In this case and under some conditions, the Pareto optimal equilibrium that would obtain under complete markets can in fact be attained under financial autarky. This is so because under this specification, the relative price of goods moves just enough to offset changes in their relative supply so that investors are perfectly insured against shocks in the economy. As a result, trade in asset is not required to reach perfect risk sharing. Another consequence is that the payoffs of the two equity assets are perfectly correlated, so that the portfolio choice of international investors is indeterminate.[22] Further, an economy with unit elasticity of substitution across goods satisfy the conditions of the no-trade theorem in **?**, so that there is no trade in equilibrium, resulting in no realistic international capital flows and no nontrivial portfolio rebalancing. Taken together, those reasons make this case clearly peculiar, and I instead focus on the more general environment in which $\theta \neq 1$, which has received less attention.

Second, the home bias in consumption, captured by parameter $\alpha > \frac{1}{2}$, is not only realistic and well-established, but also turns out to be the core driver of economic outcomes in the

[21]For instance, $\theta = 1$ in Pavlova and Rigobon (2007, 2008, 2010), Colacito et al. (2018), Maggiori (2017), or Colacito et al. (2018), among others.

[22]I discuss this case in more detail throughout Section 3. Interestingly, another consequence of the equity assets being perfectly correlated is that markets are technically dynamically incomplete when the investors can only trade the two equity assets and a bond, as discussed in Ehling and Heyerdahl-Larsen (2015). Despite this fact, investors are perfectly insured via changes in the relative price of goods.

model. It is indeed responsible for the differing patterns of asset returns and, by introducing hedging motives, is also a prime determinant of portfolios. Because it is symmetric, the home bias in consumption leads to a natural and fundamental degree of heterogeneity between investors, even in the absence of other differences. This is one of the reasons why the allocation of wealth is neither constant nor purely monotonically related to the relative supply of goods even under perfect risk sharing. In addition, this heterogeneity makes the hedging motives different across international investors, and is therefore responsible for part of the differential tilt in their portfolios that ultimately explains their individual bias towards holding more of the domestic or foreign asset. How those hedging motives interact with the impact of the allocation of wealth is an important dimension in this environment. Lastly, without home bias, the investors of both countries would consume identical baskets, so that their relative price would be constant and equal, i.e. the real exchange rate would be constant and equal to unity. This would therefore prevent the analysis of any phenomenon involving the real exchange rate, which is key quantity in an international context.

The domestic investor allocates a share $w_{h,t}$ of her wealth to the domestic equity asset, earning an expected risk premia $\mu_{R,t} - r_t$, a share $w_{f,t}$ to the foreign equity asset, earning $\mu_{R*,t} - r_t$, and the rest $(1 - w_{h,t} - w_{f,t})$ to the international bond. She uses the proceeds to purchase her desired basket of consumption $c_t \equiv C_t/W_t$, at price P_t. In other words, she chooses her consumption and portfolios to maximize (3) subject to the following budget constraint

$$\begin{aligned}\frac{dW_t}{W_t} &= \left(r_t + w_{h,t}\left(\mu_{R,t} - r_t\right) + w_{f,t}\left(\mu_{R*,t} - r_t\right) - P_t c_t\right) dt \\ &\quad + \left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right)^T d\vec{Z}_t\end{aligned} \tag{5}$$

The impact on the budget constraint of the introduction of imperfect financial integration and labor income of the form considered in this paper is discussed in Section 2.4. Finally, to complete the definition of the optimization problem, the investor is subject to a standard transversality condition, and W_0 is given. Note also that $W_t \geqslant 0$.

The problem solved by the foreign investor is similar, with the important observation that all parameters are in principle allowed to differ. It is shown in Appendix A.3. Again, note that even in the absence of any other difference, the consumption of each investor is always biased towards their own local good, which makes them heterogeneous in all cases.

2.3. Equilibrium and state variables

The definition of the equilibrium is standard: (1) investors solve their optimization problems by taking aggregate stochastic processes as given, and (2) goods and equity markets clear. It is shown in Appendix A.4. The bond market clears by Walras's law, which gives rise to the following useful relationship: $W_t + W_t^* = Q_t + Q_t^*$. In words, world wealth has to be held in the form of the two equity assets in aggregate.

Stationary recursive Markovian equilibrium Most importantly, the equilibrium can be recast as a stationary recursive Markovian equilibrium in which all variables of interest are expressed as a function of a pair of state variables $X_t \equiv (x_t, y_t)'$, whose dynamics are also solely a function of X_t. x_t is the wealth share of the domestic investor, and y_t is the relative supply of the domestic good.[23] Both are defined below.

The characterization of the solution as a system of coupled algebraic and second-order partial differential equations is the focus of Section 3. For now, let us discuss the intuition behind both state variables. Note that an additional variable, which is not a state variable but is useful throughout, is z_t, the ratio of the home equity price to world wealth. It captures the weight of the domestic asset in the market portfolio, and it can be shown that

$$z_t \equiv \frac{Q_t}{Q_t + Q_t^*} = \left(1 + \left(\frac{F_t}{F_t^*}\right) q_t \left(\frac{1 - y_t}{y_t}\right)\right)^{-1} \tag{6}$$

Wealth share The wealth share of the domestic investor captures the allocation of worldwide wealth, and is therefore a measure of the average investor in the international economy. It is defined as

$$x_t \equiv \frac{W_t}{W_t + W_t^*} \tag{7}$$

Importantly, the wealth share is not constant, even under perfect risk sharing. This is due to the fact that preferences are not log, contrary to a large subset of the literature, and to the presence of home bias in consumption. In addition, the wealth share is not solely a monotonic function of current fundamentals, so that it is required as an additional state variable. This comes from the combination of heterogeneity, introduced if nothing else by the home bias in consumption, and recursive preferences. The intuition is that the wealth share captures Negishi weights, which are time-varying in this case, as discussed among others in Dumas et al. (2000), Anderson (2005), or Colacito and Croce (2011, 2013). One of the advantages of characterizing the solution directly as a function of the wealth share is that the method remains valid even in cases in which risk sharing is imperfect, markets are incomplete, and the characterization of a solution using the Pareto weights chosen by a fictitious planner is no longer necessarily possible.

Relative supply The relative supply of the domestic good is the domestic output share. It captures the effect of current fundamentals and is defined as

$$y_t \equiv \frac{Y_t}{Y_t + Y_t^*} \tag{8}$$

[23]Formally, this is shown using a guess and verify approach like e.g. in Gârleanu and Panageas (2015). The variables of interest are: $\{c_{h,t}, c_{f,t}, c^*_{h,t}, c^*_{f,t}, w_{h,t}, w_{f,t}, w^*_{h,t}, w^*_{f,t}, \mu_{R,t}, \mu_{R*,t}, r_t, F_t, F^*_t, p_t, p^*_t, P_t, P^*_t, q_t, \mathcal{E}_t\}$.

This variable has been the focus in one form or another of a large part of the international portfolio choice literature.[24] As I discuss in Section 3.4, it is for instance closely related to the impact of real exchange rate hedging, as emphasized e.g. in Coeurdacier (2009), and Coeurdacier and Rey (2013), although the mapping is not one-to-one. An appeal of my framework is to analyze the effect of this variable in a context with more general preferences, various specifications, as well as globally throughout the state space, instead of having to rely on local approximation methods around a particular point like as been common in the literature. In addition, the interaction of the hedging of y_t with the impact of the wealth share x_t constitutes an important new dimension. I discuss those elements in detail in Section 3.

Note that because $W_t \geqslant 0$ and $Y_t \geqslant 0$, x_t and y_t are both evolving in the bounded interval $[0, 1]$. This has the advantage that solving for unknown functions on a bounded domain is numerically more stable. Conceptually, as x_t gets closer to either of the boundaries, the economy converges (continuously) to a natural one-investor environment. As y_t gets closer to either of the boundaries, the economy converges to a one-good one-equity asset economy, but this has consequences in terms of marginal values of wealth as the investors still want to consume both goods.

Throughout, I focus on the solution to the decentralized, i.e. Radner, equilibrium instead of relying on the social planner's problem. When markets are complete and risk sharing is perfect, both solutions must coincide. In Appendix D.1, I show that this is indeed the case for instance under symmetric CRRA preferences in which the elasticity of intertemporal substitution is inversely related to the risk aversion.[25] Solving for the planner solution can be extended to recursive preferences in a Markovian setting, following Dumas et al. (2000). However, I stick to the study of the decentralized economy because part of the appeal of the framework is that it remains valid even in cases in which the usual planner solution can no longer necessarily be used, such as with imperfect financial integration or even incomplete markets. This will also prove useful as the framework is extended in several directions, some of which presented in Section 4.5. An additional benefit is to put the solution closer to observables, which could prove interesting from the perspective of bringing the model to the data.

Proving existence and uniqueness of the equilibrium has proven elusive in multi-agent contexts, and this paper is no exception. Recent progress has been made in situations with potentially dynamically complete markets[26], as in Anderson and Raimondo (2008), or with

[24] Note that the ratio involves *quantities* of the two different goods. This poses no particular theoretical issue and is used because it simplifies the characterization of the equilibrium. This definition is a monotonic transformation of Y_t^*/Y_t: $y_t \equiv (1 + Y_t^*/Y_t)^{-1}$, which ensures that the state variable evolves in the bounded interval $[0, 1]$. Y_t^*/Y_t has the clear interpretation of the output of the foreign good per unit of domestic good. An economic intuition is that one compares the economy to the symmetric point in which relative prices are $q_t = \mathcal{E}_t = 1$.

[25] I also check the solution with Monte-Carlo simulations.

[26] A securities market is potentially dynamically complete if the number of securities with non-colinear payoffs is equal to one plus the number of risk factors in the form of Brownian motions to be spanned.

complete markets with a full set of Arrow-Debreu securities, as in Hugonnier et al. (2012). However, the introduction of multiple goods complicates the matter further, for instance because markets can become dynamically incomplete even if the number of assets should technically be sufficient to span risks. Together with an older literature, those multiple-good contexts are discussed in **?**, and Ehling and Heyerdahl-Larsen (2015). In addition, most of those contributions rely on the use of the Pareto efficient allocation obtained from a planner when markets are complete, and such proofs for decentralized Radner equilibria, in particular under general market structures including imperfect financial integration and incomplete markets, are currently unavailable. This is particularly true for multiple-good contexts, which can fall victim to the type of issues described in the famous Hart (1975) example. In summary, there is to date no possibility of proving the existence and uniqueness of the equilibrium in a general context such as the one in this paper, and one should humbly keep this caveat in mind as we proceed.

2.4. Additions

Together with preferences that are general and potentially heterogeneous even beyond the home bias in consumption, the framework accommodates two important additions: imperfect financial integration, and labor income.

Market structure and imperfect financial integration In the environment described so far, markets are potentially dynamically complete in the sense of Anderson and Raimondo (2008), i.e. the number of securities is at least one more than the number of independent sources of uncertainty and they can therefore span all risk. Even though the introduction of multiple goods could actually render markets dynamically incomplete, the assumption that the elasticity of substitution across goods θ is different from one, i.e. that the aggregator is not Cobb-Douglas, limits that possibility in practice (**?**Ehling and Heyerdahl-Larsen, 2015). This is because as θ differs from one, the payoffs of the two equity assets are not perfectly correlated so that they can indeed span both sources of uncertainty and the portfolio choice between them is well-defined. In short, in this setup, risk sharing is perfect, markets are complete in the usual sense, and the decentralized equilibrium is Pareto efficient and corresponds to the planner's problem.

An aspect that is important in practice however, is that international markets are likely to be imperfectly integrated. This can come from a number of frictions – informational, legal, technical –, with the result that the risk sharing between international investors is likely to be imperfect. This aspect is particularly relevant in this context because as investors have a more difficult time sharing risks with one another, the worldwide allocation of wealth among them, which is captured by x_t and is an important new dimension in this paper, is likely to have a more significant impact on economic outcomes.

To study this friction, I introduce imperfect financial integration in a parsimonious way as a tax on foreign dividends, adapting Bhamra et al. (2014) to a non-log two-good context with home bias.[27] The assumption allows me to study the effect of a range of financial integration degrees without having to take a specific stance on the source of the underlying imperfections. This tax is meant to encompass the wide array of frictions mentioned above – be they legal, technical, informational, or otherwise – that prevent investors from freely participating in foreign markets. As shown in Gârleanu et al. (2020), models with investment taxes constitute an equivalent, but substantially simpler, way to capture a rich set of impediments to financial trade. Note that the spanning condition above is still verified in that the number of securities is still one more than the number of independent sources of uncertainty. As a result, investors still individually face markets that are dynamically complete. However, the opportunity sets that they face are now different due to the tax that differentially affects the assets for each of them, so that the equilibrium need not be Pareto efficient and the usual planner solution that has been popular in the international finance literature cannot be used.[28] Relatedly, the stochastic discount factors of the two international investors are no longer perfectly correlated, and risk sharing is therefore imperfect. The latter is the phenomenon of interest here, with respect to the broader international finance and international portfolio choice literature. From a more general perspective, recall that I solve for the decentralized equilibrium of this economy, so that the framework is readily set to tackle a wide range of market structures including incomplete market settings. This will prove useful when tackling a number of promising extensions of the framework.[29]

In practice, each investor pays a tax τ (τ^*) on foreign dividends. For instance, the domestic investor only receives a dividend $(1-\tau)p_t^* Y_t^*$ per share of the foreign equity asset (of which she holds $w_{f,t} W_t / Q_t^*$) because she pays $\tau p_t^* Y_t^*$ as a tax. As a result, the risk premium on the foreign asset faced by the domestic investor and therefore appearing in her budget constraint becomes $\mu_{R^*,t} - r_t - \tau F_t^*$, while the risk premium on the domestic asset faced by the foreign investor and appearing in his budget constraint becomes $\mu_{R,t} - r_t - \tau^* F_t$. This highlights the role of dividend yields in driving the effect of the tax, a point that is important in practice as discussed in Section 3.5. The amount of tax collected from one

[27] Cf. also the seminal contribution of Basak and Gallmeyer (2003), who study a dynamic asset pricing model with asymmetric dividend taxation and a unique risky asset in a one-country one-good setting. The friction could also affect the diffusion of asset returns, which could be adapted to capture more specifically effects about information and uncertainty in the spirit of Gehrig (1993). I leave this exploration for future research.

[28] Cf. Basak and Gallmeyer (2003) for details. In a simpler context, e.g. with log preferences, one good and no home bias, one could potentially use a weaker notion of a social planner to solve the equilibrium by introducing time-varying Pareto weights, à la Cuoco and He (1994) or Basak and Cuoco (1998).

[29] As a stark example, I consider the case in which market integration is so limited that investors can only trade their local equity asset and a bond. In that case, the spanning condition is no longer satisfied and markets are incomplete. I omit those results in the interest of space, but they are available upon request. Another way to naturally include incomplete markets could be for the tax on foreign dividends to be time-varying, or to introduce idiosyncratic labor income as in Kaplan et al. (2018), or capital risk as in Brunnermeier and Sannikov (2014, 2015). Assessing the impact of those extensions is an exiting avenue for future research.

investor is rebated lump-sum to the other investor, so as not to distort decisions further. The exact details of this rebate do not make material difference, as discussed in Bhamra et al. (2014). In terms of budget constraints, the domestic investor receives an additional $w_{h,t}^*(1-x_t)\tau^* F_t/x_t$ per unit of wealth each infinitesimal period, while the foreign investor receives $w_{f,t} x_t \tau F_t^*/(1-x_t)$.

Labor Income Another aspect that has been analyzed in the literature and that can have a large impact on portfolios is labor income. Although I only touch upon it briefly in Section 3.5, this is also captured in the framework as a constant share of the output of each tree in the spirit of Baxter and Jermann (1997). Specifically, a share δ (δ^*) of the output of each tree is paid as labor income, while the remainder $1-\delta$ ($1-\delta^*$) is paid as dividends. In turn, this means that the dividend yields of the equity assets become $F_t \equiv (1-\delta)p_t Y_t/Q_t$ and $F_t^* \equiv (1-\delta^*)p_t^* Y_t^*/Q_t^*$, while the budget constraints have an additional term, $\delta F_t z_t/((1-\delta)x_t)$ and $\delta^* F_t^*(1-z_t)/((1-\delta^*)(1-x_t))$, for the domestic and foreign investor respectively.[30]

A more general specification of labor income could be an interesting extension, and is left for future research. It could for instance take the form of a time-varying share of output, as in Coeurdacier and Gourinchas (2016), and could naturally give rise to incomplete markets, or more realistic hedging terms in portfolios. The discussion in Section 3.5 and Appendix A.8 provides additional details.

2.5. Computation of the equilibrium

Section 3, which follows, characterize all variables of interest as a function of the state variables, $X_t = (x_t, y_t)'$, and a set of unknown functions $\mathcal{G} \equiv \{J_t,\, J_t^*,\, F_t,\, F_t^*,\, q_t,\, w_{h,t},\, w_{f,t}\}$.[31] Due to the stationary recursive Markovian structure of the equilibrium, those unknown functions are themselves solely functions of X_t, and are determined by a set of coupled algebraic and second-order partial differential equations. Before describing those in the next section, let me say a brief word about the numerical approach.

Each of the unknown function $g : [0,1]^2 \to \mathcal{D}^g \in \mathbb{R}$ in $\mathcal{G}$ is approximated using projection methods based on Chebyshev polynomials and orthogonal collocation. Details are provided in Appendix C. Let us simply discuss a few characteristics of the approach.

First, as with many continuous-time approaches, projection methods provide a global solution throughout the state space. This is in sharp contrast to a large subset of the international portfolio choice literature that has historically focused on local approximations

[30] z_t, the ratio of the home equity price to world wealth, is updated accordingly: $z_t \equiv Q_t/(Q_t + Q_t^*) = \left(1 + \left((1-\delta^*)/(1-\delta)\right)\left(F_t/F_t^*\right) q_t \left(1-y_t\right)/y_t\right)^{-1}$.

[31] J_t, J_t^* are introduced in Section 3.2 and capture (an increasing monotonic transformation of) the marginal values of wealth of each investor. In addition, as a point of notation, for any function g, g_t simply denotes $g(X_t)$, not the time-derivative of g (which is zero because the model is stationary due to infinite horizon).

in neighborhoods of specific points.[32] The use of a global solution approach instead makes it possible to study economic outcomes throughout the state space, which is particularly relevant in contexts in which variables e.g. portfolios are strongly state-dependent such as here. In addition, the use of a global solution method is important in cases in which the evolution of the variables of interest are very non-linear throughout the state space as can be the case in this context when investors become more risk averse, and even more importantly when markets become less perfectly integrated or investors become more heterogeneous. Such a method is also particularly adapted when there is no particularly well-suited point around which to perform a local approximation, such as a steady state. This is the case in my framework due to the specification of outputs as geometric Brownian motions but more importantly is also typically true in international contexts with incomplete markets. Finally, a global method will prove crucial when different types of constraints on the investor are introduced, a natural element that I plan to include in future research as discussed in Section 4.5.

Second, projection methods are also well-suited to contexts with multiple state variables in which other approaches like finite-difference methods become rapidly computationally too costly.[33] More generally, the addition of new state variables, as will naturally happen with the planned extensions discussed in Section 4.5, pose conceptually no difficulty. To be sure, computationally, traditional projection methods also are very much subject to the curse of dimensionality and scaling the number of state variables will prove limited using standard Chebyshev polynomials so that methods able to handle higher-dimensional cases will be required.[34] One such method consists in naturally extending the concept of projection approaches, but to replace the Chebyshev polynomials in the approximation by neural networks, which are designed specifically to handle high-dimensional contexts. I am developing these "projection methods via neural networks" for continuous-time models in Sauzet (2020b), and I discuss them in slightly more details in Section 4.5 and Appendix E.3.

[32] A typical point around which the local approximation is performed in the literature is the deterministic or risky steady state if it is well-defined, or the symmetric point in the middle of the state space (it would correspond to $X_t = (1/2, 1/2)$ in my context). A notable exception is Rabitsch et al. (2015), who use a global method. However, their framework is cast in discrete time so that the authors do not discuss the underlying drivers of portfolios, in particular hedging demands, the conditional (time-varying) moments of asset returns, as well as the conditional (time-varying) state variable dynamics. Under a set of assumptions, local methods could be used to study an economy further in the state space, cf. for instance Mertens and Judd (2018). However, such methods remain difficult to use in an international portfolio context due to the portfolio indeterminacy that arises in the corresponding deterministic economy.

[33] The method currently developed in Hansen et al. (2018) could potentially help from that perspective.

[34] Finer ways to construct the Chebyshev polynomials and corresponding grids, such as complete polynomials or Smolyak's algorithm, can help. Ultimately however, they are also limited.

3. Characterization of the Equilibrium

I characterize the equilibrium to the economy presented in Section 2, both theoretically and by showing the resulting solution. The solution sheds light on the importance of the allocation of wealth, which impacts the economy as a state variable capturing the average international investor, and as a pricing factor against which investors hedge. Home bias in equity holdings can obtain in the setup either when the elasticity of substitution across goods is low, or due to imperfect financial integration provided that the elasticity of intertemporal substitution is moderate. In all cases, the bias in portfolio holdings is amplified by the hedging of the wealth share risk. In addition, portfolios as well as other variables strongly vary throughout the state space, emphasizing the importance of the global solution. The quantitative impact of the allocation of wealth increases with imperfect financial integration, and investor heterogeneity. In other words, "capital is back" in this international context: the allocation of wealth matters.

The parameters used to obtained the equilibrium functions are set according to the symmetric calibration of Assumption 1, unless otherwise specified. The elasticity of substitution across goods, θ, and the elasticity of intertemporal substitution, ψ, are of particular interest. The former determines the sign of portfolio hedging in the baseline, while the latter is a strong determinant of whether imperfect financial integration has a large quantitative impact. The values for both are consistent with standard recent estimations in Imbs and Méjean (2015) for θ, and Schorfheide et al. (2018) for ψ, and I discuss their effects in detail in the main text. Risk aversion is on the high side, although reasonable, to generate more realistic risk premia and Sharpe ratios. Other parameters are standard, and details are provided in Appendix A.7. All proofs are relegated to Appendix B.

Assumption 1 (Symmetric baseline calibration)**.** *Unless otherwise specified, the results presented in this section are obtained under the following calibration:*

- *Risk aversion:* $\gamma = \gamma^* = 15$,
- *Elasticity of intertemporal substituion:* $\psi = \psi^* = 2$,
- *Home bias in consumption:* $\alpha = \alpha^* = 0.75$, *numéraire basket:* $a = 1/2$,
- *Elasticity of substitution between goods:* $\theta = \theta^* = 2$,
- *Discount rate:* $\rho = \rho^* = 1\%$,
- *No labor income:* $\delta = \delta^* = 0$,
- *Fully integrated financial markets:* $\tau = \tau^* = 0$,
- *Output:* $\mu_Y = \mu_Y^* = 2\%, \sigma_Y = (4.1\%, 0)^T, \sigma_Y^* = (0, 4.1\%)^T$ *(no fundamental correlation).*

3.1. Evolution of the state variables

Due to the Markovian nature of the equilibrium, the laws of motion of the state variables underlie the dynamics of the economy. They are summarized in Proposition 1. The relative supply y_t being exogenous, I focus the discussion on the endogenous state variable x_t.

Proposition 1. *Laws of motion for the wealth share x_t, and relative supply y_t:*

$$\frac{dx_t}{x_t} \equiv \mu_{x,t}dt + \sigma_{x,t}^T d\vec{Z}_t \tag{9}$$
$$\frac{dy_t}{y_t} \equiv \mu_{y,t}dt + \sigma_{y,t}^T d\vec{Z}_t$$

where:

$$\begin{aligned}
\mu_{x,t} &= (w_{h,t} - z_t)(\mu_{R,t} - r_t) + (w_{f,t} - (1 - z_t))(\mu_{R*,t} - r_t) \\
&\quad + (F_t z_t + (1 - z_t)F_t^*) - P_t c_t + \left(\frac{\delta}{1-\delta}\right) F_t \left(\frac{z_t}{x_t}\right) + \tau^* F_t \left(\frac{z_t}{x_t} - w_{h,t}\right) - \tau F_t^* w_{f,t} \\
&\quad - ((w_{h,t} - z_t)\sigma_{R,t} + (w_{f,t} - (1 - z_t))\sigma_{R*,t})^T (z_t\sigma_{R,t} + (1 - z_t)\sigma_{R*,t}) \\
\sigma_{x,t} &= ((w_{h,t} - z_t)\sigma_{R,t} + (w_{f,t} - (1 - z_t))\sigma_{R*,t}) \\
\mu_{y,t} &= (1 - y_t)(\mu_Y - \mu_{Y*}) - (1 - y_t)(\sigma_Y - \sigma_{Y*})^T (y_t\sigma_Y + (1 - y_t)\sigma_{Y*}) \\
\sigma_{y,t} &= (1 - y_t)(\sigma_Y - \sigma_{Y*})
\end{aligned}$$

The drift of the wealth share, $\mu_{x,t}x_t$, is shown in Figure F.24 in Appendix. It reflects the different forces impacting the budget constraints of both investors – returns on portfolios, consumptions, labor income, and the tax on foreign dividends that capture imperfect financial integration–, and drives the dispersion of the wealth share in equilibrium.

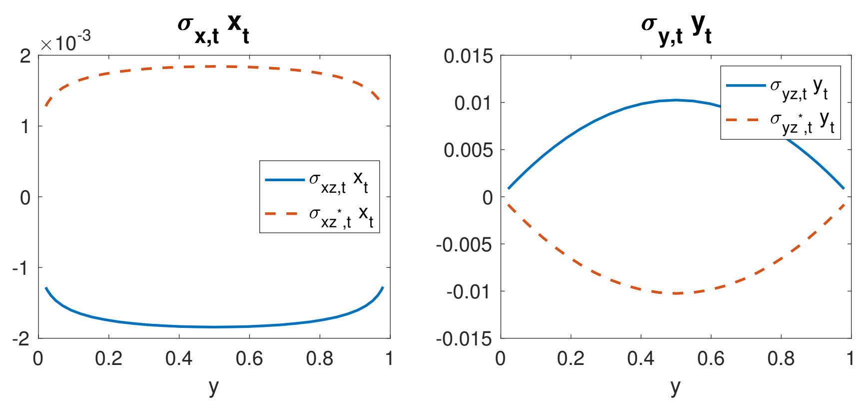
Of more importance because it will impact portfolios, the diffusion of the wealth share, $\sigma_{x,t}x_t$, is shown in Figure 1, together with that of the relative supply, $\sigma_{y,t}y_t$. To fix ideas, the figure shows both terms of each diffusion as a function of the relative supply, when the allocation of wealth is symmetric ($x_t = 1/2$).[35] This type of representations as a function of one of the state variables will be used throughout the paper when they ease the interpretation.[36] The main observation is that $\sigma_{xz,t}x_t$ is negative throughout the state space, so that a positive shock to domestic output, $dZ_t > 0$, leads the wealth share of the domestic investor to *de*crease. Except when markets are imperfectly integrated as discussed in 3.5, this is true for any calibration and reflects the interaction of a number of underlying mechanisms that I discuss in the following sections. Namely, the domestic investor invests more in the asset that has high payoffs when her marginal value of wealth is high, which occurs when the relative supply of her preferred (domestic) good is low, i.e. y_t low. When goods are good substitutes, broadly $\theta > 1$, the returns on the foreign asset are higher in this situation,

[35] Figure F.25 in Appendix shows the diffusion terms of x_t as a function of both state variables, and highlights that they vary a lot throughout the state space also in the x_t dimension. In particular, they are largest around $x_t = 1/2$, the point at which a switch occurs in which of the investors dominates the world economy.

[36] Note that this is only for the purpose of visualization. The equilibrium is still solved as a function of both state variables in all cases.

$\sigma_{Rz,t} > \sigma_{R^*z,t}, \sigma_{Rz^*,t} < \sigma_{R^*z^*,t}$, so that the domestic equity portfolio exhibits a foreign bias compared to the market portfolio, $w_{h,t} - z_t < 0, w_{f,t} - (1 - z_t) > 0$. This corresponds to the standard calibration of θ in the baseline. When goods are poor substitutes instead, $\theta < 1$, the domestic asset has higher returns in this this situation, $\sigma_{Rz,t} < \sigma_{R^*z,t}, \sigma_{Rz^*,t} > \sigma_{R^*z^*,t}$, so that the domestic equity portfolio exhibits a home bias, $w_{h,t} - z_t > 0, w_{f,t} - (1 - z_t) < 0$. The combination of those sets of facts yields the negative loading of the wealth share on domestic shocks, $\sigma_{xz,t}x_t < 0$, and the positive loading on foreign shocks, $\sigma_{xz^*,t}x_t > 0$, in all cases. Those patterns in turn will determine the sign of the hedging of wealth risk on portfolios that I discuss in Section 3.4.

Figure 1: Diffusion terms for the state variables



Notes: Based on the symmetric calibration of Assumption 1. The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding representations as a function of both variables: Figure F.25.

The resulting equilibrium distribution of the state variables is shown in Figure F.3 in Appendix. The sign of the diffusion terms discussed above is reflected in the strong negative relationship between the wealth share of the domestic investor, which tends to decrease for positive domestic output shocks, and the relative supply of the domestic good, which increases in that case. The dispersion of the wealth share around this broad negative relationship is driven by the drifts and increases with the elasticity of intertemporal substitution. In this framework, the wealth share is time-varying, as soon as we move away from the log case ($\gamma = \psi = 1$), and is not purely determined by current fundamentals, y_t, as soon as we move way from the CRRA case ($\psi \neq 1/\gamma$) and introduce recursive preferences, or as we introduce imperfect financial integration. Even though the dispersion remains modest in the baseline, it increases significantly as markets become imperfectly integrated, and as investors become more heterogeneous, a point I discuss in Section 3.5.

3.2. Marginal values of wealth, goods prices, and risk sharing

I now turn to the marginal value of wealth of the investors, a quantity that underly many decisions in the economy. To characterize them, note that due to the homotheticity of preferences, the value functions of the investors can be expressed as

$$V(W_t, x_t, y_t) = \left(\frac{W_t^{1-\gamma}}{1-\gamma}\right) J(x_t, y_t)^{\frac{1-\gamma}{1-\psi}} \tag{10}$$

$$V^*(W_t^*, x_t, y_t) = \left(\frac{W_t^{*1-\gamma^*}}{1-\gamma^*}\right) J^*(x_t, y_t)^{\frac{1-\gamma^*}{1-\psi^*}}$$

Because W_t and W_t^* mostly have an impact in levels, the relative marginal values of wealth of the two investors, which are obtained as the derivative of the value functions with respect to wealth, are primarily driven by the powers of J_t and J_t^*. In the remainder of the text, I therefore sometimes refer loosely to J_t and J_t^* as (monotonic transformations of) the marginal values of wealth.[37] J_t and J_t^* are important economic objects in that they drive a large part of the dynamics of the stochastic discount factors of the two investors, which in turns determine portfolios, asset prices, and other economic decisions. Indeed, in this context, stochastic discount factors can be expressed as[38]

$$\xi_t \equiv \xi_0 \exp\left\{\int_0^t \left(\Theta_1 P_u^{1-\psi} J_u + \Theta_2\right) du\right\} W_t^{-\gamma} J_t^{\frac{1-\gamma}{1-\psi}} \tag{11}$$

$$\xi_t^* \equiv \xi_0^* \exp\left\{\int_0^t \left(\Theta_1^* P_u^{*1-\psi^*} J_u^* + \Theta_2^*\right) du\right\} W_t^{*-\gamma^*} J_t^{*\frac{1-\gamma^*}{1-\psi^*}}$$

The evolution of J_t and J_t^* are governed by two Hamilton-Jacobi-Bellman equations, summarized in Proposition A.2 in Appendix.[39] Figure 2 shows the result for the domestic investor in the baseline calibration as a function of both fundamentals (y_t), shown on the horizontal axis, and the wealth share of the domestic investor (x_t), shown as different curves.[40] Results are symmetric for the foreign investor.

The intuition is as follows, and will be at the core of the differential tilt in the portfolio of each investor. As the domestic good becomes relatively scarce, i.e. as y_t decreases, the marginal value of consumption for the domestic investor increases given that she wishes to consume more of this good that she prefers, but cannot due to its limited supply. Following

[37] For instance, in the baseline calibration, $(1-\gamma)/(1-\psi) > 0$, and this is an *increasing* monotonic transformation. In terms of notation, recall that J_t, J_t^* simply denote $J(X_t), J^*(X_t)$, with $X_t = (x_t, y_t)'$.

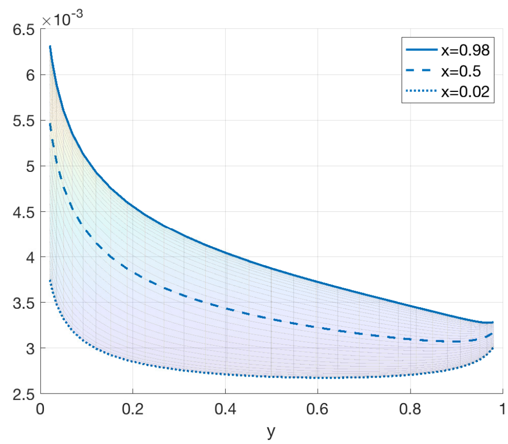
[38] Constants $\Theta_1, \Theta_2, \Theta_1^*$ and Θ_2^* are provided in Appendix B.2.

[39] These are two coupled second-order partial differential equations. The boundary conditions are the natural ones that result as the geometric drifts and diffusion terms of x_t and y_t converges to 0 when x_t and y_t approach 0 and 1, respectively.

[40] A number of corresponding three-dimensional representations are also available in Appendix F.9 for the reader to whom they make the visualization more straightforward.

a standard envelope argument, the marginal value of wealth follows the same pattern and J_t therefore increases as y_t decreases, a phenomenon that occurs for any value of the wealth share and is the main driver of the J_t. On the other hand, the marginal value of wealth increases with x_t, reflecting the fact that as she becomes dominant in the world economy, the domestic investor gets closer to holding the market portfolio, is thus unable to diversify risks as much with the foreign investor that becomes increasingly small, and is therefore relatively worse-off. From a macroeconomic standpoint, those patterns are consistent with the marginal value of wealth of the investor increasing as the price of her preferred (domestic) good rises, which happens as its relative supply y_t is low, or as the domestic investor owns a large share x_t of world wealth.

Figure 2: Domestic marginal value of wealth (J_t)



Notes: Based on the symmetric calibration of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding three-dimensional representation: Figure F.33.

From the Hamilton-Jacobi-Bellman equations in A.2, a first set of first-order conditions yield expressions for consumptions, summarized in Proposition A.3, which emphasize once again the underlying role of J_t and J_t^*: $c_t \equiv C_t/W_t = P_t^{-\psi} J_t$, and $c_t^* \equiv C_t^*/W_t^* = P_t^{*-\psi^*} J_t^*$. In the interest of space, details are shown in in Appendix A.6 together with the corresponding figures, which are as expected. Combining with market-clearing conditions, one obtains Equation (12) for the terms of trade q_t, shown in Proposition 2.

Proposition 2. *The terms of trade, $q_t = q(X_t)$, solves the following non-linear equation:*

$$q_t = S_t^{1/\theta} \left(\frac{y_t}{1 - y_t} \right)^{1/\theta} \tag{12}$$

where:

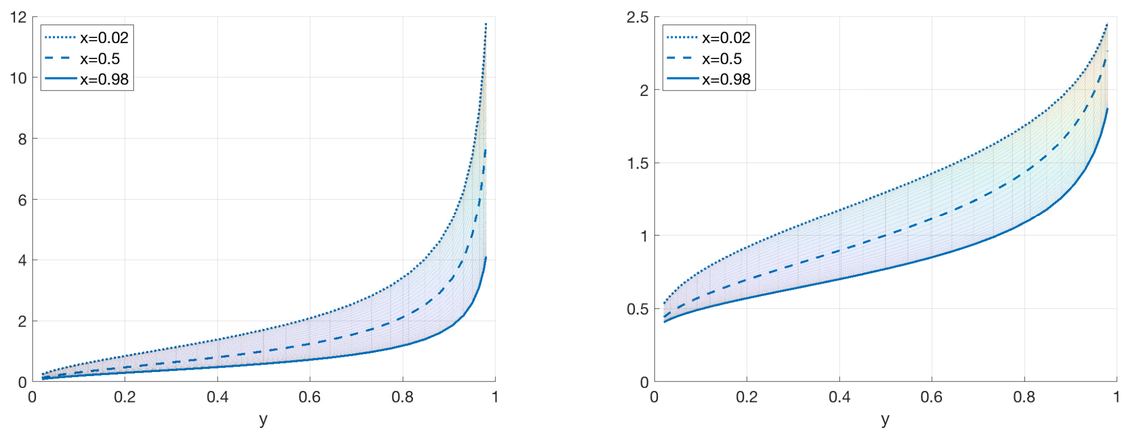
$$S_t = \frac{(1-\alpha) P_t^{\theta-\psi} J_t x_t + \alpha P_t^{*\theta-\psi^*} J_t^* (1 - x_t)}{\alpha J_t x_t P_t^{\theta-\psi} + (1-\alpha) P_t^{*\theta-\psi^*} J_t^* (1 - x_t)}$$

Prices $p_t, p_t^*, P_t, P_t^*, \mathcal{E}_t$ follow from the definition of the numéraire and Proposition A.3, and are shown in Proposition A.4.

This expression is the equivalent of Coeurdacier (2009)'s in this generalized framework, and emphasizes two main determinants of relative prices: the relative supply of the goods, captured by $y_t/(1-y_t) = Y_t/Y_t^*$, and the relative demand for them, captured by S_t. The latter is akin to a transfer effect in the spirit of Keynes and Ohlin and depends on the allocation of wealth in the world economy as well as on the marginal values of wealth of both investors. The corresponding q_t is shown in Figure 3 together with the real exchange rate $\mathcal{E}_t$.

Figure 3: Relative prices

(a) Terms-of-trade ($q_t \equiv p_t^*/p_t$) (b) Real exchange rate ($\mathcal{E}_t \equiv P_t^*/P_t$)



Notes: Based on the symmetric calibration of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Consistent with findings in Coeurdacier (2009), who focuses on local approximations around the symmetric point $X_t = (1/2, 1/2)$, both relative prices are strongly related to the relative supply of goods, with the terms of trade worsening ($q_t \uparrow$) and the real exchange rate depreciating ($\mathcal{E}_t \uparrow$) as the domestic good becomes relatively more abundant. However, in this global framework and even around the symmetric point, the allocation of wealth in the international economy also plays a role, albeit more muted. Specifically, as the domestic investor gathers a large share of world wealth, her preference for the domestic good puts upward pressure on its price, which results in an improving terms of trade ($q_t \downarrow$) and appreciated real exchange rate ($\mathcal{E}_t \downarrow$). The introduction of the wealth share in this international portfolio choice context therefore makes the link between relative supply and relative prices less direct, so that even though the hedging of relative supply will broadly capture the hedging of real exchange rate risk, which has been the focus of the literature so far, the hedging of wealth share risk will also play a role.

Beyond relative prices themselves, which drive relative consumption decisions, the relative dividends between the two equity assets is of particular interest. They are shown in Figure F.13 and are obtained as

$$\frac{p_t^* Y_t^*}{p_t Y_t} = q_t \left(\frac{1 - y_t}{y_t} \right) = S_t^{\frac{1}{\theta}} \left(\frac{y_t}{1 - y_t} \right)^{\frac{1-\theta}{\theta}} \tag{13}$$

Contrary to relative prices, which are impacted only quantitatively by the calibration of parameters, relative dividends can flip direction. In particular, when goods are poor substitutes for one another (broadly when $\theta < 1$[41]), the transfer effect due to relative demands is so large that relative dividends on the foreign asset *in*crease when the relative supply of the foreign good decreases, as shown in Panel (a) of Figure F.13. On the other hand, in the case in which goods are better substitutes (e.g. $\theta = 2 > 1$), relative dividends move in the same direction as the relative supply of the good underlying the payoffs of each asset, as is more standard and consistent with recent estimations of the elasticity of substitution such as that in Imbs and Méjean (2015). This switch in direction is consequential because it determines which of the asset has a higher payoff as a function of relative supply and will therefore be a prime determinant of how the home bias in consumption translate into portfolios. Note that in both cases, the relative dividends of the foreign asset also decrease as the wealth share increases, consistent with the preference of the domestic investor for the domestic good that puts an upward pressure on the relative price of the domestic good as she becomes dominant in the world economy. This effect is more muted in the baseline calibration however.

Finally, under the assumption that there are not tax on foreign dividends so that risk sharing is perfect, the stochastic discount factors of both investors are perfectly correlated and we can derive in this environment a generalized version of the Backus-Smith condition of Backus and Smith (1993) and Kollmann (1995). This condition, shown in Proposition 3, emphasizes that the real exchange rate is not only determined by relative consumption, as in the usual CRRA case, but also depends on relative wealth and the marginal values of wealth of international investors. Section 3.5 discusses the case of imperfect financial integration in which we deviate from this condition.

[41] Coeurdacier (2009) shows that the exact value at which the switch occurs is in fact a non-linear function of all parameters. The author shows it in the CRRA case and at the symmetric point, but his findings are likely to persist in the framework of my paper with recursive preferences and globally. In practice, the switch is still close to $\theta = 1$ however, the case on which part of the seminal contribution of Cole and Obstfeld (1991) focuses and at which the CES aggregator of goods becomes Cobb-Douglas. In this case, relative dividends are constant and the two equity assets are perfectly correlated so that the portfolio choice is indeterminate.

Proposition 3 (Generalized Backus-Smith condition)**.** *Under symmetric recursive preferences and perfect risk sharing*

$$\mathcal{E}_t = \phi^{\frac{1}{\gamma\psi}} \exp\left\{\int_0^t \frac{1}{\gamma\psi}\left(\Theta_1(P_u^{1-\psi}J_u - P_u^{*1-\psi}J_u^*)du\right)\right\}\left(\frac{C_t^*}{C_t}\right)^{-1/\psi}\left(\frac{J_t^*}{J_t}\right)^{-\frac{1-1/(\gamma\psi)}{1-\psi}} \tag{14}$$

Constant Θ_1 *is provided in Appendix B.2, and* ϕ *is the relative Pareto weight of the two investors.*

3.3. Asset Prices

Starting with first moments, Proposition 4 presents the formulae for the expected risk premia on both equity assets, which are composed of three terms.

Proposition 4. *The expected risk premia on the equity assets are given by*

$$\begin{aligned}\mu_{R,t} - r_t = {} & \gamma_t \sigma_{R,t}^T \{z_t\sigma_{R,t} + (1-z_t)\sigma_{R^*,t}\} \\ & - \gamma_t\sigma_{R,t}^T\left\{x_t\left(\frac{1}{\gamma}\right)\left(\frac{1-\gamma}{1-\psi}\right)\sigma_{J,t} + (1-x_t)\left(\frac{1}{\gamma^*}\right)\left(\frac{1-\gamma^*}{1-\psi^*}\right)\sigma_{J^*,t}\right\} \\ & + \gamma_t\left(\frac{1-x_t}{\gamma^*}\right)\tau^* F_t\end{aligned} \tag{15}$$

$$\begin{aligned}\mu_{R^*,t} - r_t = {} & \gamma_t \sigma_{R^*,t}^T \{z_t\sigma_{R,t} + (1-z_t)\sigma_{R^*,t}\} \\ & - \gamma_t\sigma_{R^*,t}^T\left\{x_t\left(\frac{1}{\gamma}\right)\left(\frac{1-\gamma}{1-\psi}\right)\sigma_{J,t} + (1-x_t)\left(\frac{1}{\gamma^*}\right)\left(\frac{1-\gamma^*}{1-\psi^*}\right)\sigma_{J^*,t}\right\} \\ & + \gamma_t\left(\frac{x_t}{\gamma}\right)\tau F_t^*\end{aligned} \tag{16}$$

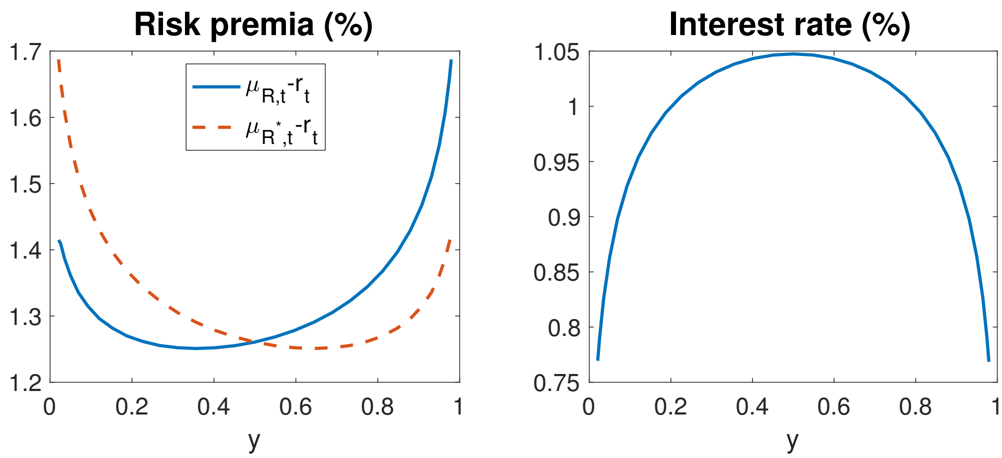
where $\gamma_t \equiv \left(\frac{x_t}{\gamma} + \frac{1-x_t}{\gamma^*}\right)^{-1}$ *is the wealth-weighted global risk aversion.*

The first term is a global component, and is driven by the covariance between each of the risky asset and world wealth.[42] Intuitively, an asset that comoves a lot with world wealth provide little diversification benefits, is therefore risky, and commands a high risk premia. The second term relates to how assets comove with the world-weighted marginal value of wealth, with the weight accounting for both differences in preferences and the allocation of wealth, and captures the fact that an asset whose payoffs are high when the world marginal value of wealth is high provides a good hedge to international investors, and therefore requires a lower risk premium (notice the negative sign). The third term is a general equilibrium effect

[42] Indeed, $z_t\sigma_{R,t} + (1-z_t)\sigma_{R^*,t}$ is the weighted-average of the diffusions of both risky assets. The weights are $z_t \equiv Q_t/(Q_t + Q_t^*)$ and $1 - z_t$ on the domestic and foreign asset respectively, which are the weight of each asset in the market portfolio. This is therefore nothing but the diffusion of world wealth.

arising when markets are imperfectly integrated due to the tax on foreign dividends, and is discussed in Section 3.5. The price of risk on all three exposures is driven by the wealth-weighted global risk aversion, γ_t, which is constant and equal to $\gamma = \gamma^*$ in a symmetric calibration, but varies with the wealth share more generally.[43]

Figure 4: Returns



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding representation as a function of both variables: F.15.

Figure 4 shows the corresponding returns as a function of fundamentals in the calibration of Assumption 1, which are representative of that for other parameter values. For both assets, the relative supply is the main determinant of expected risk premia, and those are driven almost exclusively by the global component. For instance, as the domestic good becomes abundant ($y_t \uparrow$), the domestic asset becomes dominant in world wealth ($z_t \uparrow$) so that the covariance of the domestic asset with world wealth increases sharply. In other words, the domestic asset provides increasingly poorer diversification benefits to international investors, is therefore riskier, and commands a higher risk premia. The pattern for the interest rate is consistent with the evolution of the world-weighted marginal value of wealth.[44]

[43]The expressions in Proposition 4 are also consistent with the expression for the price of risk κ_t, obtained from the fact that $d\xi_t/\xi_t = -r_t dt - \kappa_t^T d\vec{Z}_t$ in the baseline. For details, and an expression of r_t (pending) and κ_t, cf. Appendix B.2.

[44]In terms of levels, the average risk premia at around 1.4% remains small. This is not surprising given the relatively muted risk aversion of $\gamma = \gamma^* = 15$, and introducing portfolio constraints and other amplification mechanisms will be interesting extensions to consider to remedy this fact. On the other hand, the levels for the interest rate and the Sharpe ratio, at around 1% and 0.44 respectively, are broadly in line with the data. Note also that the pattern for risk premia are consistent with those for the underlying dividend yields, F_t, F_t^*, shown in Appendix, even though those depend more strongly on parameters. Dividend yields are also related to the marginal values of wealth of both investors by the

Note that the risk premia on the foreign asset also ultimately increases as y_t gets close to 1. This reflects the fact that even though both investors have a preference towards their local good, they still desire both in their consumption basket given that the goods are not perfectly substitutable. As one of the good becomes increasingly rare, the demand from both investors combined with a low supply put a significant upward pressure on the price of that good, so that the returns on that asset are driven up at the same time as those on the asset for which the relative supply becomes large. This phenomenon increases in magnitude as goods become more difficult to substitute (lower θ), and is also reflected in the conditional covariance of returns discussed below.

In the baseline, the impact of the wealth share on returns remains muted, as seen in Figure F.15, even though the impact on Sharpe ratios is more noticeable.[45] This impact grows significantly with imperfect risk sharing and investor heterogeneity as I discuss in Sections 3.5 and 4. Qualitatively, an increase in the wealth share of the domestic investor yields an increase in the risk premium on the domestic asset, and a decrease in the risk premium on the foreign asset. In the baseline in which goods are good substitutes, this occurs because the domestic (foreign) asset is a poor (good) hedge for the domestic investor. Indeed, the payoffs of the domestic asset for instance are large when her marginal value of wealth is low, which occurs primarily when her preferred (domestic) good is rare. Those patterns of risk premia as a function of the wealth share are reversed however when the goods are poor substitutes ($\theta < 1$) because relative dividends become inversely related to the relative supply in that case, as observed previously and shown in Figure F.13.

Let us now turn to second moments.[46] Figure 5 shows the diffusion terms for the returns on both assets in the baseline calibration, as well as the (instantaneous) conditional covariance and correlation of returns.

For a change, I focus on the foreign asset. In the baseline calibration, the diffusion term corresponding to the foreign shock ($\sigma_{R^* z^*,t}$) is larger for most of the state space. While intuitive, given that the physical output underlying this asset Y_t^* loads mostly on this shock, the result once again hinges on the degree of substitutability across goods due to the fact that asset payoffs also depend on goods prices. When goods are good enough substitutes like in the baseline ($\theta = 2 > 1$), Section 3.2 showed that the relative dividends on the foreign asset, $p_t^* Y_t^* / (p_t Y_t)$, increases when the relative supply y_t decreases, so that the returns on the foreign asset loads more on the foreign shock. When goods are poor substitutes however ($\theta < 1$), the relative dividends on the foreign asset increases when y_t *in*creases, due

following expression:

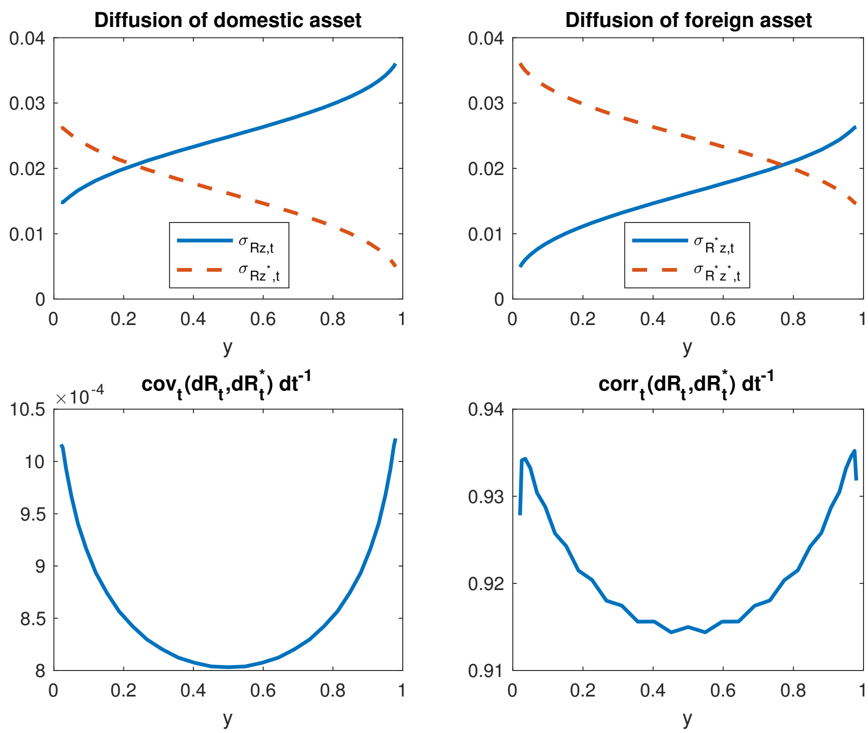
$$P_t^{1-\psi} J_t x_t + P_t^{*1-\psi^*} J_t^* (1 - x_t) = \left(\frac{1}{1-\delta}\right) F_t z_t + \left(\frac{1}{1-\delta^*}\right) F_t^* (1 - z_t) \tag{17}$$

[45] This is confirmed by computing the elasticities of risk premia and Sharpe ratios with respect to both state variables.

[46] As a side note, one of the strengths of the continuous-time framework is that it allows to express all *conditional* moments, both first and second, directly as a function of state variables.

to a strong effect on goods prices coming from consumer demand, so that the foreign asset ultimately loads more on the *domestic* shock.[47] The former case is consistent with standard modern estimations of the elasticity of intertemporal substitution, e.g. in Imbs and Méjean (2015), so it remains my baseline, but the sign of the loadings will ultimately be responsible for the direction of the portfolio bias that obtains in equilibrium.

Figure 5: Second moments of returns



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the domestic good, which captures fundamentals. Three-dimensional representation: Figure F.32.

Beyond these differences, it is noteworthy that regardless of θ, both returns load on both shocks. This is so despite the fact that the output of each good only loads on its local Brownian shock, i.e. $\sigma_{Yz*} = \sigma_{Y*z} = 0$. For instance, in the baseline, the diffusion term corresponding to the loading of foreign returns on the domestic shock ($\sigma_{R*z,t}$) is positive and large throughout. In fact, as the domestic good becomes dominant, the latter becomes larger than the loading on its own shock! This pattern is driven by changes in goods prices, as shown in the decomposition in Appendix F, and emphasizes that the conditional moments of asset returns vary significantly throughout the space. Economically, this highlights the

[47]In the limit case in which $\theta = 1$, as discussed in Cole and Obstfeld (1991), the payoffs on both assets are perfectly correlated so that the portfolio choice between them is indeterminate.

strong international contagion taking place through asset markets: a shock on the output of a given country has a large impact on the returns of the tree of the other country, and can therefore impact investors worldwide beyond its impact on goods markets.

The above is also reflected in the evolution of the covariance and correlation of returns. First, and most striking, both are large, again despite no fundamental correlation in output. Those findings are consistent with those in Pavlova and Rigobon (2007), who focus on a log-Cobb-Douglas case, and Bhamra et al. (2014), who focus on a log-CES case with no home bias. They are, however, reinforced in this generalized environment. For instance, the correlation is above 0.9 throughout the state space and reaches as high as 0.94 depending on the state of the economy, well above that in Bhamra et al. (2014), who find a correlation around 0.5 in a one-good specification (with a fundamental correlation of 0.5), and slightly above 0.6 in a two-good specification with no home bias and $\theta = 2$. Similarly, the magnitude is significantly larger than in the one-good specification of Chabakauri (2013), for whom the correlation does not increase beyond 0.5, and sharply decreases towards the boundaries. This emphasizes the impact of a two-good environment with home bias for asset pricing, in which comovements in good prices have a large effect on comovements in returns. It also highlights the quantitative difference made by using different calibration of preferences. For instance, the average level of correlation increases with the home bias in consumption α, risk aversion γ, and the elasticity of intertemporal substitution ψ, and has an inverted U-shape pattern as a function of the elasticity of substitution across goods, with a maximum around $\theta = 1$ in which both assets are perfectly correlated.

Second, covariance and correlation are also time-varying and change a lot throughout the state space, an aspect that the global solution make possible to characterize. Specifically, as one of the good becomes abundant, returns increasingly comove, consistent with the evolution of diffusion terms in Figure 5. This phenomenon is the manifestation for second moments of the pattern that was also observed for expected returns. As seen in Figure F.32, the correlation of returns itself also vary strongly with the wealth share, even in the baseline, and has a saddle shape: it is significantly larger around $x_t = 1/2$, the point of the state space at which the switch in which investor dominates the economy occurs. For instance, for $y_t = 1/2$, $\mathrm{corr}_t(dR_t, dR_t^*)dt^{-1} = 0.879$ for $x_t \to 0$ or 1, but 0.915 for $x_t = 1/2$. The correlation also displays a slight asymmetry and reaches its minimum around $x_t = 0.35$ for low y_t and $x_t = 0.65$ for high y_t.

Taken together, those result emphasizes that the benefits of diversification provided by each asset, as measured by the comovement of their returns, depend a lot on the calibration, and are also inherently state-dependent in this context.

3.4. Portfolios

From the Hamilton-Jacobi-Bellman equations in Proposition A.2, a second set of first-order conditions yield the optimal portfolios in Proposition 5. Those are typical Merton (1973)-type portfolios and are composed of two pieces.

The first one is common to both investors when markets are perfectly integrated ($\tau =$

$\tau^* = 0$) up to differences in risk aversion. It corresponds to the myopic portfolio that would be chosen by a one-period mean-variance investor, and is driven by the risk premia on both assets, normalized by volatilities.

The second piece is a hedging term, absent with log or myopic preferences, which captures the way investors tilt their portfolios to insure against changes in the state of the economy, captured by $X_t = (x_t, y_t)'$. They do so by overweighting assets whose payoffs are large when they find it most valuable, i.e. when their marginal values of wealth are high, so that hedging terms are governed by the covariance between marginal values of wealth, J_t, J_t^*, and risky returns. First, investors hedge the relative supply risk, y_t. Because the relative supply is a strong driver of the relative prices of goods, this aspect is intimately related to the hedging of real exchange rate risk that has been the focus of a large part of the literature: investors form their portfolios by hedging against changes in the relative prices of the goods that they desire to consume. Yet, as was visible in Figure 3, the mapping between relative supply and relative prices, although strong, is not one-for-one and is also impacted by the repartition of wealth across investors. The framework in this paper allows to disentangle those different channels: in general equilibrium, investors hedge not only against relative supply changes, i.e. changes in the physical quantity of the goods, but also against changes in their share of wealth. The latter, which had so far not been emphasized in the international portfolio choice literature, matters both because it has an impact on relative prices, but also as it captures the extent to which investors are able to share and diversify risks with one another.

Overall, the common term drives the broad pattern of the portfolios of both investors throughout the state space, while the hedging term captures how international investors differentially deviate from this broad pattern. Hedging terms are therefore a prime variable of interest in an international context.

Proposition 5. *The home and foreign portfolios are given by*

$$\begin{pmatrix} w_{h,t} \\ w_{f,t} \end{pmatrix} = \frac{1}{\gamma} \left(\Sigma_t^T \Sigma_t\right)^{-1} \left\{ \begin{pmatrix} \mu_{R,t} - r_t \\ \mu_{R^*,t} - r_t - \tau F_t^* \end{pmatrix} + \left(\frac{1-\gamma}{1-\psi}\right) \Sigma_t^T \left(\frac{J_{x,t}}{J_t} x_t \sigma_{x,t} + \frac{J_{y,t}}{J_t} y_t \sigma_{y,t} \right) \right\}$$
$$b_t = 1 - w_{h,t} - w_{f,t} \tag{18}$$

$$\begin{pmatrix} w_{h,t}^* \\ w_{f,t}^* \end{pmatrix} = \frac{1}{\gamma^*} \left(\Sigma_t^T \Sigma_t\right)^{-1} \left\{ \begin{pmatrix} \mu_{R,t} - r_t - \tau^* F_t \\ \mu_{R^*,t} - r_t \end{pmatrix} + \left(\frac{1-\gamma^*}{1-\psi^*}\right) \Sigma_t^T \left(\frac{J_{x,t}^*}{J_t^*} x_t \sigma_{x,t} + \frac{J_{y,t}^*}{J_t^*} y_t \sigma_{y,t} \right) \right\}$$
$$b_t^* = 1 - w_{h,t}^* - w_{f,t}^* \tag{19}$$

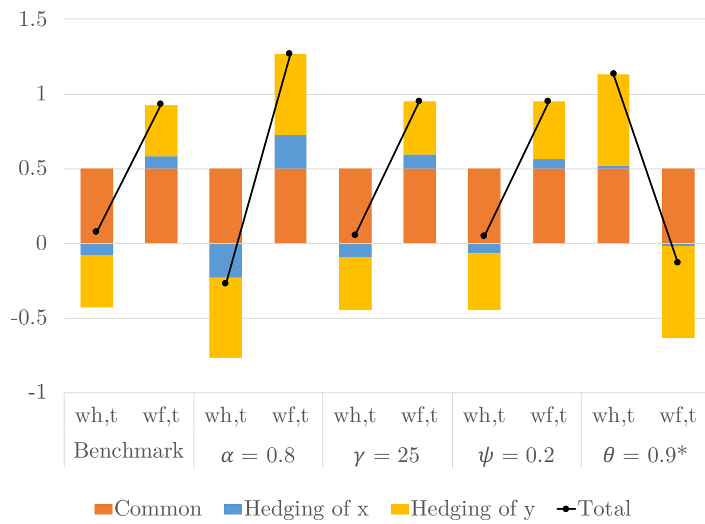
where $\Sigma_t \equiv \begin{bmatrix} \sigma_{R,t} & \sigma_{R^*,t} \end{bmatrix}$.

What do portfolios look like in practice in this context? I start by discussing average portfolios.

The international portfolio choice literature has for the most part considered so-called zero-order (i.e. steady-state) portfolios. These are constant and replicate locally complete markets in a small neighborhood of the symmetric point of the state space by using a second-order approximation of portfolio equations, and a first-order approximation to other equations.[48] Even though one of the main advantages of the global method I introduce in this paper is to break away from those low-order local approximations as I discuss below, I first investigate patterns at the symmetric point $X_t = (1/2, 1/2)$ to ease comparison with existing work. I start by focusing on the hedging of relative supply risk, to make the parallel with the hedging of real exchange risk that has been most discussed.

Average portfolios are strongly impacted by the specification of preferences. To see this, Figure 6 shows the weights allocated to the domestic and foreign equity assets in the domestic portfolio, $w_{h,t}$, $w_{f,t}$, at $X_t = (1/2, 1/2)$ for various calibrations. Figures F.10, F.11 and F.12 also provide additional details.[49]

Figure 6: Equity portfolio at $X_t = (1/2, 1/2)$



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except for the specified parameters. * For $\theta = 0.9$: $\gamma = 15, \psi = 1/\gamma, \alpha \approx 0.58$ (further calibrations ongoing). The figure shows portfolios when both the allocation of wealth (x_t) and the relative supply (y_t) are symmetric, $X_t = (1/2, 1/2)$.

The most important dimension is once again the elasticity of substitution across goods, which can flip the bias in portfolio holdings. Recall that due to the home bias in consumption, the marginal value of wealth of the domestic investor increases when the domestic good becomes rare (y_t decreases): she would like to consume more of her preferred good but cannot. Symmetrically, the marginal value of wealth of the foreign investor decreases, given his preference for the foreign good that becomes abundant. As a result, the domestic investor

[48]For examples of this approach, cf. Coeurdacier (2009), Tille and van Wincoop (2010), Devereux and Sutherland (2011), Coeurdacier and Rey (2013), and Coeurdacier and Gourinchas (2016), among others.

[49]Note that in this symmetric calibration with perfect risk sharing, the international bond is not traded. I come back to this aspect in further sections.

values an asset that pays in those conditions, while the foreign investor does not. What is the asset that pays most when the domestic good becomes rare? When goods are good substitutes like in the baseline ($\theta = 2 > 1$), Sections 3.2 and 3.3 showed that relative dividends and returns are positively related to relative supply, so that this is the foreign asset. The domestic investor therefore overweights the foreign asset in her portfolio, while the foreign investor symmetrically overweights the domestic asset. In other words, portfolios exhibit a foreign bias in equity holdings. When goods are poor substitutes instead ($\theta < 1$), relative dividends and returns are negatively related to relative supply so that the payoffs of the *domestic* asset gets larger when y_t decreases, and it gets overweighted in the portfolio of the domestic investor and underweighted in that of the foreign investor. Portfolios therefore exhibits a home bias in equity holdings. Those patterns are typically consistent with the hedging of real exchange risk that has been one of the focus of the literature, as discussed e.g. recently in Coeurdacier (2009). Indeed, an asset that pays well when the relative supply of an investor's preferred good is low is also an asset that pays well when the relative price of that good is high. It is therefore valued by that investor, and overweighted in their portfolio.

The discussion suggests that a first explanation for why portfolios might be biased towards domestic assets empirically could be that the goods produced by different countries are poor substitutes, and is consistent with findings in Heathcote and Perri (2002), Kollmann (2006), Corsetti et al. (2008).[50] This calibration however can be called into questions for three reasons. First, even though it is the subject of some debate in the literature, standard modern estimations of θ typically put it above one, with Imbs and Méjean (2015)'s popular estimate in the range of $[4, 6]$. Values above one are also consistent with a large body of empirical work in international trade. Second, the case of $\theta < 1$ also has a number of counterfactual predictions: (i) growth is immesirizing, i.e. the output of a country at market value decreases for a positive supply shock so that a positive domestic shock mostly benefits the foreign country, and (ii) the introduction of other realistic aspects of international trade and macroeconomics, such as trade costs, leads to an even worse foreign bias in equity holdings in this situation, as discussed in Coeurdacier (2009). Third, even though a low θ could yield the right direction in terms of portfolios, the home bias obtained as a result is in fact too extreme for reasonable calibrations of the parameters. This aspect, hinted at in Coeurdacier (2009), is confirmed in my general setup: e.g. even for α as low as 0.58, significantly below usual calibrations such as the baseline of $\alpha = 0.75$, each investor *shorts* the foreign asset at the symmetric point ($w_{f,t} = w^*_{h,t} - 13\%$) in order to allocate more than 100% of their wealth to the local asset when $\theta = 0.9$. For all those reasons, I stick to the standard case of $\theta > 1$ as my baseline. To turn the foreign bias in equity holdings that obtains into a home bias like in the data, I will instead rely on the other (plausible) channel that I introduce in this environment: imperfect financial integration. I discuss this aspect in Section 3.5.

What about the effect of wealth share hedging? When risk sharing is perfect, the hedging of wealth share risk turns out to reinforce the bias that emerges from the hedging of relative supply. To see this, let us focus on the baseline calibration and consider a negative shock to

[50] The impact of this assumption is also discussed in Tille (2001) and Coeurdacier (2009).

domestic output ($dZ_t < 0$). In that case, the wealth share of the domestic investor increases, as was discussed in 3.1.[51] This in turn leads her marginal value of wealth to increase, given that it is more difficult for her to diversify risk, so that she values an asset that pays in those conditions. Concurrently, such a shock tends to decrease relative output y_t as well as the relative dividends on the domestic asset, provided that goods are good enough substitutes, so that its payoffs decreases relative to those on foreign equity. This is so because the mild upward pressure on the price of the domestic good coming from the increase in the size of the domestic investor in the economy is not enough to compensate the downward pressure due to the lower supply. As a consequence, the domestic asset therefore does not pay off in a situation where it is would be valuable, which leads the domestic investor to tilt her portfolio further away from the domestic asset. The phenomenon is reversed in cases of low substitutability of goods, so that wealth share hedging reinforces the home bias that obtains in that case. Quantitatively, the impact of the hedging of x_t remains muted in the baseline, but grows significantly as investors become more heterogeneous and as markets become imperfectly integrated so that sharing risk is more difficult across investors. I discuss those aspects below and in Section 3.5.

The impact of both hedging terms taken together is significant and the foreign bias in equity holdings that obtains in the baseline is large: at the symmetric point, the domestic investor allocates $w_{h,t} = 93\%$ of her wealth to the foreign asset, and only $w_{f,t} = 7\%$ to the domestic asset, compared to the 50-50% split consistent with the common term.[52] The bias is reinforced as the risk aversion increases (Figure F.11), making investors more sensitive to risks in the economy, while (very) mildly reduced as the elasticity of intertemporal substitution increases (Figure F.12). For both, the hedging of wealth risk grows in importance, even though it remains mostly muted compared to that of relative supply risk. The bias in portfolios is also strongly reinforced as the home bias in consumption increases (Figure F.10). This makes the investors have a stronger preference towards their local good, which strengthens the hedging of y_t, but also make them more heterogeneous, which strengthens the effect of the wealth share even more. As a result, the impact of wealth share hedging is strongly reinforced and becomes as large as that of y_t. This heightened impact of the wealth share risk is a theme that will come back when I study further heterogeneity as well as imperfect financial integration in Section 3.5, and for the application of Section 4. For large values of α, the portfolio bias can become extreme. For instance, with $\alpha = 0.85$, a value that is still lower than the types of values used more recently in the literature, $w_{h,t} = -175\%$ and $w_{f,t} = 275\%$, in words, the domestic is willing to severely short the domestic asset to lever up the share of her wealth that she allocates to the foreign asset. Similarly, the *home* bias is strongly reinforced when $\theta < 1$. Overall, those results are broadly consistent with findings in Coeurdacier (2009), even though the author focused on a CRRA case in which the risk

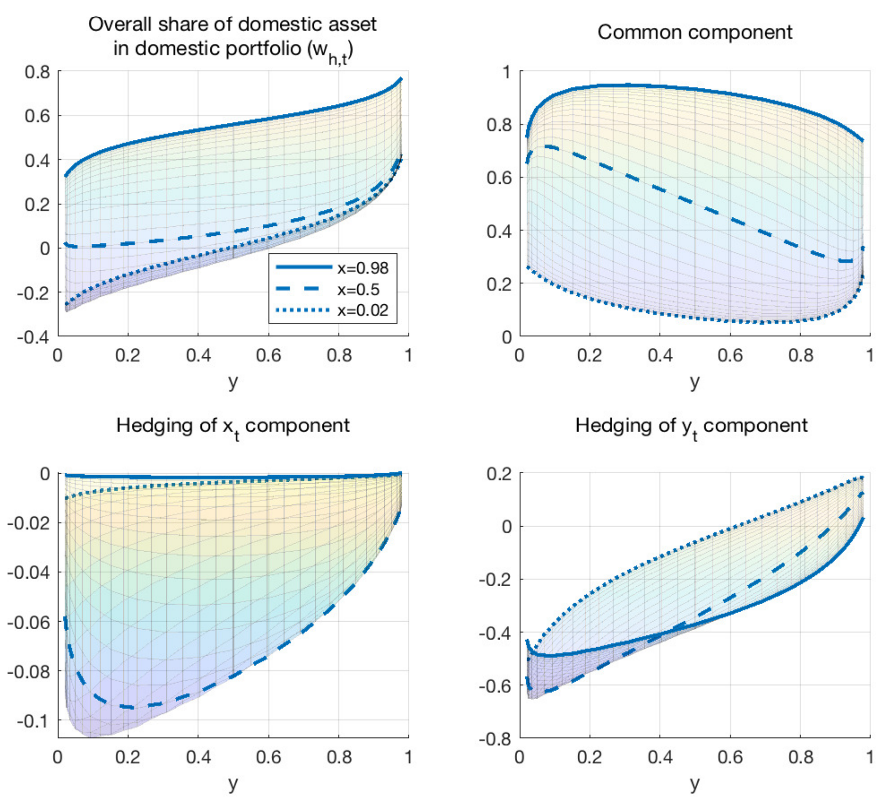
[51] Recall that this was due to the combination of the signs of $w_{h,t} - z_t < 0$, $w_{f,t} - (1 - z_t) > 0$, which obtain mostly from the hedging of relative supply in the baseline, and of the patterns of the diffusion of risky returns, $\sigma_{R,t}, \sigma_{R*,t}$.

[52] Again, this would also be the case with a low θ so that the resulting home bias in equity holdings would still be counterfactually large, with the foreign asset not being invested in or even being shorted.

aversion and elasticity of substitution are inversely related to one another and in which, most importantly, the wealth share did not play a role of its own. In addition, the non-linearities that obtain as heterogeneity increases render the use of a global method particularly important as compared to first and second-order local approximations. Beyond magnitudes, hedging terms are also important in that they drive the differential tilt in the portfolios of the two international investors.

What happens beyond the symmetric point? Being able to study portfolios and other variables not just for $X_t = (1/2, 1/2)$ but for any point of the state space is one of the breakthroughs allowed by the global method in this paper. Conceptually, this is a natural way forward given that even for the symmetric point, the hedging of the state variables is fundamentally about what is happening outside of this point, i.e. about dynamics throughout the state space, which cannot be visualized and studied with local low-order approximations and constant portfolios but that the method here suddenly make completely visible.

Figure 7: Components of the domestic portfolio



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding three-dimensional representations: Figure F.34.

Figure 7 shows the components of the weight of the domestic asset in the domestic portfolio in the baseline calibration. What happens at $X_t = (1/2, 1/2)$ was discussed before, but the picture reveals that portfolios and their components vary substantially with the state of the economy. For instance, the domestic strongly shorts the domestic asset, $w_{h,t} = -30\%$, when the relative good is rare and when her share of world wealth is small, while she allocates a large portion of her portfolio to it , $w_{h,t} = 80\%$, when the domestic good is abundant and her share of wealth large. This picture is quite contrasted to the $w_{h,t} = 7\%$ that is picked by the domestic investor at the symmetric point.

This dependence on the state of the economy comes both from the common component, which drives the overall shape of both portfolios, and from hedging terms. The relative supply of course has a strong impact on each component, for instance with the hedging of the relative supply becoming much stronger as the domestic good becomes rare. But most importantly, the picture suggests a second important role for the wealth share in addition to its impact as a pricing factor that is hedged: its role as a state variable, which captures the average investor in the world economy in a given instant. Because this average investor looks very different according to whether she most resembles the domestic or foreign investor, the wealth share has a strong direct impact on the common component, highlighting its effect on risk premia and the conditional variance-covariance matrix of returns, as well as on the hedging of the wealth share itself, which is largest around $x_t = 1/2$, the point around which the dominant investor in the world economy switches and at which the volatility of the wealth share is largest.

What about the bias in portfolios in this global context? To study it, it is no longer sufficient to compare $w_{h,t}$ and $w_{f,t}$, the weight of each asset in the domestic portfolio. Indeed, as the state of the economy evolves, the share of each asset in the market portfolio also changes compared to the 50-50% split that obtains at the symmetric point. To study this question, I therefore compute a portfolio bias measure towards the domestic equity, HB_t, and a portfolio bias measure towards the foreign equity, FB_t. Those measures have the added benefit that they are closer to those that have been used empirically, for instance in Coeurdacier and Rey (2013).[53] They are defined as the share of domestic equity in the domestic equity portfolio ($w_{h,t}/(w_{h,t} + w_{f,t})$) divided by its share in the market portfolio (z_t), and the share of foreign equity in the domestic equity portfolio ($w_{f,t}/(w_{h,t} + w_{f,t})$) divided by its share in the market portfolio ($1 - z_t$), i.e.

$$HB_t \equiv \frac{w_{h,t}/(w_{h,t} + w_{f,t})}{z_t} \quad \text{and} \quad FB_t \equiv \frac{w_{f,t}/(w_{h,t} + w_{f,t})}{1 - z_t} \tag{20}$$

[53] The fact that $w_{h,t}$ and $w_{f,t}$ vary throughout the state space could mean that we could potentially explain the fact that $w_{h,t}$ is above $w_{f,t}$ in practice, provided that the world economy is in some particular part of the state space. However, in the baseline calibration, the foreign bias is quantitatively so stark that the regions of the state space in which $w_{h,t} > w_{f,t}$ are small. Cf. Figure XX. In addition, the proper way to study the home bias is to compare portfolio weights to the market portfolio, as done in the rest of the paper.

Those measures are shown in the bottom two panels of Figure F.14 for the baseline calibration, and their equivalent for the foreign investor, HB_t^*, FB_t^*, are defined analogously.[54] They paint an even starker picture than the components discussed above: portfolios vary substantially with the state of the economy not only in terms of the weights themselves, but also in terms of how biased they are. For instance, as she becomes dominant in the world economy, and even though equity prices adjust accordingly, the domestic investor has to get closer to holding the market portfolio, i.e. both HB_t and FB_t converge to one. When her share of world wealth diminishes however, and if in addition the relative supply of the domestic good becomes rare, she shorts the domestic asset in a magnitude that is particularly extreme when compared to the market portfolio (-1.2), while she levers up and invest about 1.6 times as much than the market portfolio on the foreign asset. Those observations are mirrored in the case in which the home bias obtains when θ is low.

Taken together, those results confirm the strong bias in equity holdings that obtained in the baseline at the symmetric point ($w_{h,t}/z_t = 0.2$ at $X_t = (1/2, 1/2)$), while emphasizing that the extent of this bias is also inherently state-dependent. Note once again that both are true for high θ like in the baseline for which a foreign bias obtains, and for low θ in which a home bias obtains. The introduction of imperfect financial integration that I discuss in the next section will therefore be important to ultimately generate plausible portfolios. In addition, those evolutions reveal that portfolios are fundamentally time-varying and strongly responding to shocks to both wealth and relative supply. Those aspects could so far not be discussed in the literature, given the main focus on zero-order constant portfolios, and local neighborhoods of the symmetric point.[55] Even though empirical facts about the time evolution of home bias measures remain for the moment elusive, given the limited length of this times series and their relative smoothness due to their low frequency (annual or less), the results in this section suggest that they are an important target for future research, as echoed in Coeurdacier and Rey (2013). To that end, the large and detailed data gathering effort undertaken for the Global Capital Allocation project of Maggiori et al. (2020) and Coppola et al. (2020) will assuredly prove invaluable.

Finally, we can revisit the impact of each component quantitatively. To do so, Table 1 decomposes the (unconditional) variance of $w_{h,t}$ into its three components: in the baseline calibration with $\alpha = 0.75$, hedging components drive 30% of the changes in portfolios, and this proportion increases to 69% as $\alpha = 0.85$.[56] This confirms the picture that emerged from

[54] Recall that $z_t \equiv Q_t/(Q_t + Q_t^*)$, i.e. it is the ratio of the home equity price to world wealth. (Home equity price is the same as home equity value given that the supply of the asset is normalized to unity.) An equivalent approach would be to compute a measure of home bias as $1 - w_{f,t}/(w_{h,t} + w_{f,t})/(1 - z_t)$ as in Coeurdacier and Rey (2013). I stick to my measure because it allows to look at both assets. Note also that when the bond is not traded like here, $w_{h,t}/(w_{h,t} + w_{f,t}) = w_{h,t}$ because $w_{h,t} + w_{f,t} = 1$.

[55] Again, as investors become more heterogeneous, portfolios also become strongly non-linear (cf. Figures F.16 and F.17), so that using a global method is also crucial from this perspective as low-order local approximations could become imprecise.

[56] Specifically, I compute each of the component of the following decomposition: $1 = \frac{\mathrm{var}(w_{h,t})}{\mathrm{var}(w_{h,t})} = \frac{\mathrm{cov}(w_{h,t}^{common})}{\mathrm{var}(w_{h,t})} + \frac{\mathrm{cov}(w_{h,t}^{hedg.x})}{\mathrm{var}(w_{h,t})} + \frac{\mathrm{cov}(w_{h,t}^{hedg.y})}{\mathrm{var}(w_{h,t})}$.

the analysis of average portfolios at the symmetric point in which the hedging components were also particularly important. Although the hedging of wealth share risk itself remains muted in the baseline, it increases significantly as investors heterogeneity increases and becomes on par with the hedging of relative supply. Perhaps most importantly and as an additional remember, even when they are quantitatively smaller and while the common component drives the broad shape of the portfolio, hedging components are conceptually responsible for the differential tilt in portfolios between domestic and foreign investors, which is often the question of interest in an international portfolio choice context. Ignoring them, or focusing on special cases such as log or myopic preferences in which hedging components are absent as as been common in part of the literature, can therefore yield significantly different portfolios.

Table 1: Variance decomposition of portfolio weights

	Common component	Hedging of x_t	Hedging of y_t	Total
$\alpha = 0.5$	-42%	0%	142%	100%
$\alpha = 0.75$	70%	7%	23%	100%
$\alpha = 0.85$	31%	35%	34%	100%

Notes: The table shows the decomposition of $w_{h,t}$, the share of the domestic asset in the domestic portfolio, into its three components. $\alpha = 0.75$ is the baseline calibration. Results are identical for $w_{f,t}$, which is equal to $1 - w_{h,t}$ in the baseline, and for the portfolio of the foreign investor.

3.5. Imperfect financial integration and investor heterogeneity

Imperfect financial integration and investor heterogeneity are two dimensions that have the potential to strongly impact the equilibrium. I study both in this international context, and show that their influence goes hand-in-hand with a reinforced effect of the allocation of wealth.

Imperfect financial integration The introduction of imperfectly integrated markets, modeled as a tax τ on foreign dividends in the spirit of Bhamra et al. (2014), impacts the economy because it prevents international investors from perfectly sharing risk with one another.[57] Because the assets that they can trade are different, due to the direct tax that each investor has to pay on them as well as a general equilibrium effect on their risk premia, the opportunity sets faced by both investors differ. As a result, even though they individually face dynamically complete markets, their stochastic discount factors are no longer perfectly

[57] In this section, I assume that $\tau = \tau^*$, i.e. that the tax on foreign dividend is symmetric. However, the framework also allows for asymmetric taxes, which can be interesting to study in realistic applications. For details on the exact formulation of those taxes in the model, cf. Section 2.4.

correlated. This has a number of consequences in terms of the evolution of their marginal values of wealth, interest rates, consumptions, and other variables, with the effect most visible on portfolios. The general specification of the model also allows us to study the impact of several dimensions of preferences, which turn out to have a strong impact on the magnitude of the effect of imperfect financial integration. I focus on the elasticity of intertemporal substitution ψ, which takes center stage.

When the elasticity of intertemporal substitution is low, $\psi = 0.2$[58], the introduction of a modest degree of imperfect financial integration is sufficient to make their respective foreign asset much less attractive to each local investor. This comes both from the fact that the overall level of the foreign risk premium as perceived by a local investor is directly decreased by the tax that has to be paid on it, $-\tau F_t^*, -\tau^* F_t$, and from the fact that the slope of the risk premia on both asset as a function of the wealth share flips sign driven by the tax as well as a modest general equilibrium effect.[59]

As a result, both international investors rapidly turn the foreign bias in equity holdings that they picked in the baseline, into a significant home bias in equity holdings. For instance, at the symmetric point $X_t = (1/2, 1/2)$, the left panel of Figure 8 shows that a tax on foreign dividends on the order of $\tau = 7$ to 10% is enough to bring the HB_t measure above 1, from 0.75 to 1.39, and the FB_t measure below 1, from 1.32 to 0.50, both reflecting a strong home bias in equity holdings compared to the market portfolio and consistent with empirical measures e.g. in Coeurdacier and Rey (2013). To get a sense of magnitude, the raw share of the domestic asset in the domestic equity portfolio increases from 42% to 78% at that point, broadly in line with the data. The fact that reasonable frictions on market integration can yield home bias in equity confirms the finding of Bhamra et al. (2014) in this general and global framework, provided that ψ is low.

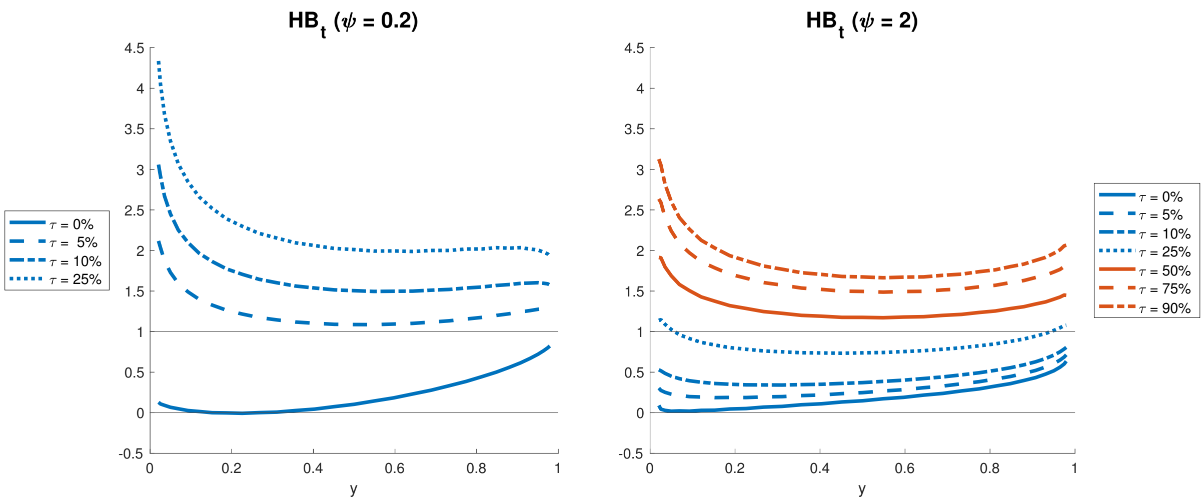
In addition, contrary to the baseline studied so far, the international bond is now traded in equilibrium (Panel (a) of Figure 10), reflecting the fact that less risk sharing can happen via the equity assets so that investors make use of the third asset. Bond trading is also strongly asymmetric. The share of wealth allocated to the bond, b_t, b_t^*, strongly decreases as the wealth share of an investor increases: an investor cannot borrow from herself when she becomes dominant in the world economy, a fact that participates in reinforcing the influence of the wealth share on portfolios. On the other hand, whether investors save or borrow using the bond is governed by the relative supply of their preferred good. For instance, the domestic investor saves using the bond ($b_t > 0$) when the domestic good is abundant. This reflects the fact that she can consume a lot of her preferred good, so that her marginal value of wealth is low. As a result, she saves some of her wealth for situations in which this is

[58] This is slightly higher than the CRRA case, $\psi = 1/\gamma \approx 0.067$, to ensure that investors still have preference for early resolution of uncertainty.

[59] This is even more visible on the Sharpe ratio on the top right panel of Figure 9, which combines the effect on the risk premia and second moments. While domestic risk premium and Sharpe ratio increased with x_t in the baseline, even with $\psi = 0.2$, they now both increase with imperfectly integrated markets, reflecting the now positive relationship between x_t and y_t discussed below.

not the case, and in which her marginal value of wealth is higher. Conversely, the domestic investor borrows as y_t decreases.

Figure 8: Domestic equity portfolio vs. market portfolio



Notes: Based on the symmetric calibration of Assumption 1, except for ψ and τ. The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the domestic good, which captures fundamentals. Effect on the foreign bias measure FB_t: Figure F.18.

The stark switch in the tilt of portfolios comes with a larger impact of the wealth share.[60] This stems in part from its reinforced direct effect as a state variable: the identity of the investor holding most of the wealth in the international economy, captured by x_t, matters more when risk sharing is imperfect because investors have a more difficult time insuring against risks in the economy. This direct impact can be observed on portfolios as well as other variables in Figure 9 for a tax of $\tau = 10\%$.[61] Second, the impact of the hedging of wealth share risk also grows markedly. Quantitatively, the variance decomposition of $w_{h,t}$ yields shares of 37%, 33%, and 35%, for the common, x_t-hedging, and y_t-hedging components, respectively. The wealth share hedging therefore plays a much larger role now on par with other components, compared to the 7% it was responsible for in the baseline.[62] Further, not only does the magnitude of the hedging of x_t changes, but so does its sign. While the hedging of fundamentals still make any investor dislike their local asset (as long as goods are realistically good enough substitutes), the hedging of x_t is now positive, meaning that

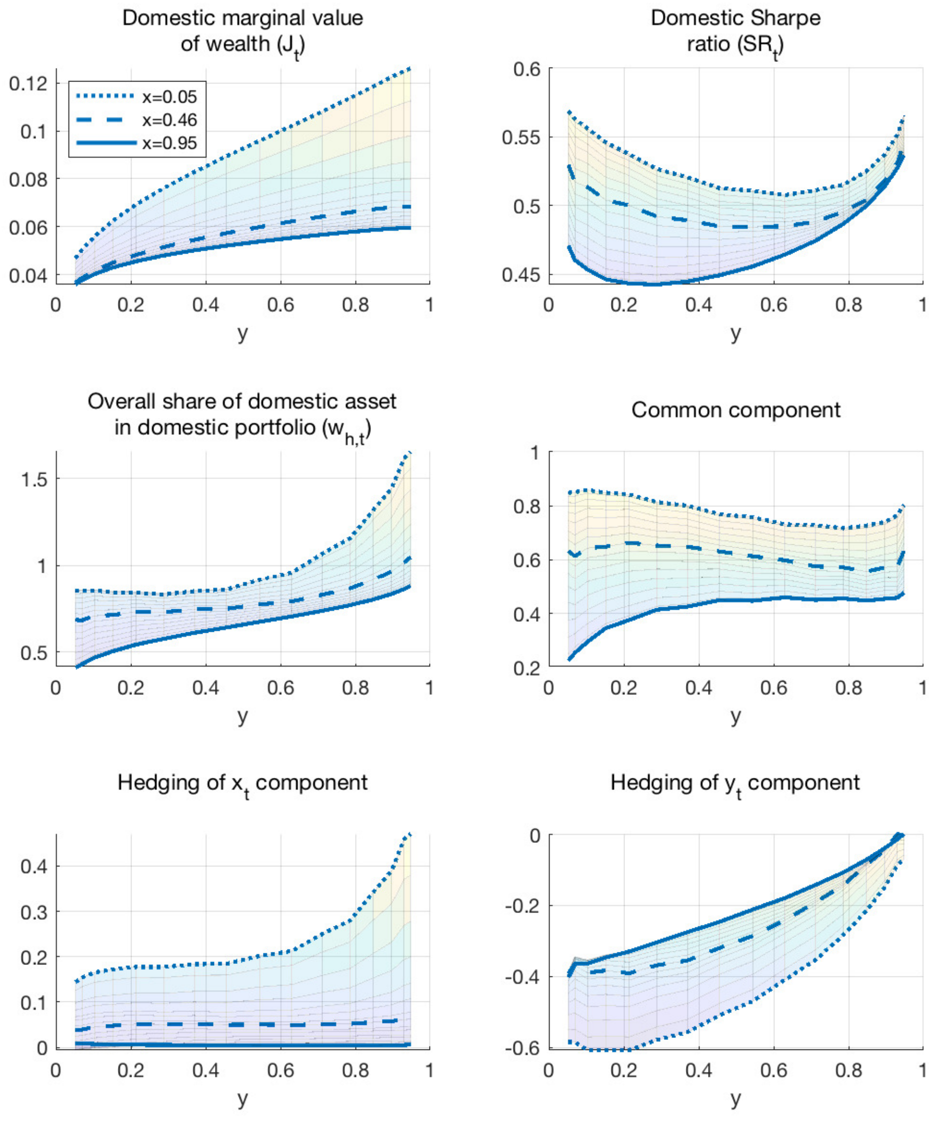
[60] As before, portfolios also vary significant with the relative supply of goods. For instance, HB_t ranges from 1.25 to above 3, depending on whether the domestic good becomes abundant ($y_t \to 1$) or scarce ($y_t \to 0$).

[61] Note that with $\psi = 0.2$, the actual marginal value of wealth, $J_t^{\frac{1-\gamma}{1-\psi}}$, is a *de*creasing monotonic transformation of J_t.

[62] For the foreign asset, those numbers are 18%, 32%, and 27%, with the remaining 23% attributed to the tax payment itself. For the baseline those were of 70%, 7%, and 23%, for both $w_{h,t}$ and $w_{f,t}$.

instead of reinforcing the foreign bias coming from y_t like it did in the baseline, it directly contributes to obtaining the home bias in equity holdings.

Figure 9: Impact of the wealth share under imperfectly integrated markets ($\tau = 10\%$) with a low elasticity of intertemporal substitution ($\psi = 0.2$)



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except that $\psi = 0.2$ and $\mu = 10\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

This happens because in this case, the loading of the wealth share on the Brownian shocks flips, with the wealth share now *in*creasing for a positive domestic output shock, so that the domestic asset provides a good hedge for changes in the allocation of wealth for the domestic investor. This flip occurs because of the overpowering effect of imperfect financial integration on portfolios: by making the domestic asset more attractive, the tax yields a home bias in equity holdings in equilibrium so that compared to the market portfolio, $w_{h,t} - z_t > 0$ and $w_{h,t} - z_t < 0$, the opposite of the baseline case. Following Proposition 1, this results in $\sigma_{xz,t} x_t > 0$, $\sigma_{xz^*,t} x_t < 0$. In words, the wealth share of the domestic investor loads positively on domestic output shocks, and negatively on foreign output shocks.[63] Finally, note that instead of being broadly symmetric around $x_t = 1/2$, the hedging of wealth share risk now tends to decrease with x_t, reflecting its larger impact on the slope of J_t closer to small values of x_t.

Turning now to the case where the elasticity of intertemporal substitution is high, $\psi = 2$, as in the (realistic) baseline calibration of Section 3, the picture changes drastically. As seen on Panel (b) and (d) of Figure 8, taxes on foreign dividends now have much more limited impact on portfolios. As an example, reasonable taxes on the order of $\tau = 7$ or 10% do not overturn the counterfactual foreign bias in equity holdings, like they did for $\psi = 0.2$, let alone bringing it to the ballpark estimate observed in practice. In fact, for this to happen even only for the symmetric point of the state space, τ has to climb to values as high as 50, 75%, or more, which are clearly implausible.

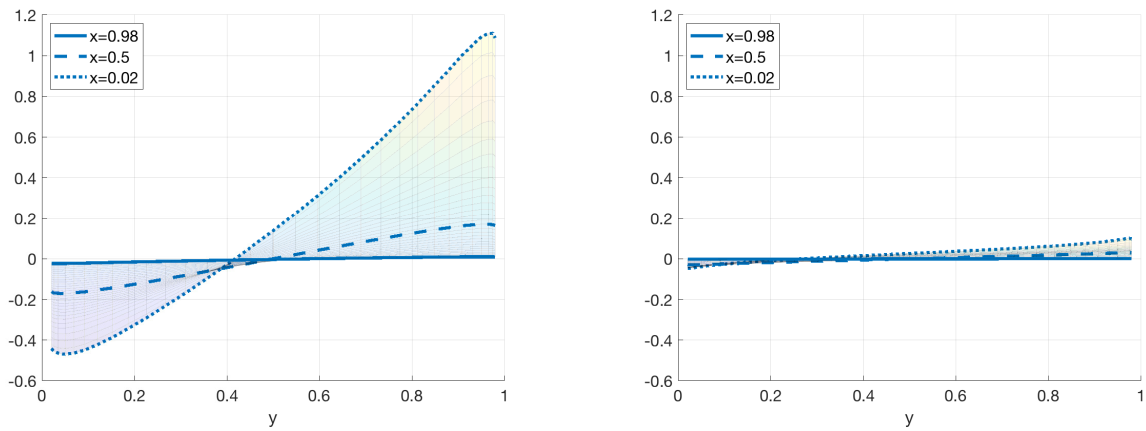
Why does the elasticity of intertemporal substitution have such a central role? This comes for a large part from its impact on the dividend yields, F_t, F_t^*, of the two equity assets. With a large ψ, substitution effects dominate so that investors value assets even when they pay far in the future. The resulting equity prices, which are nothing but the present value of the streams of dividend paid by the assets discounted with the appropriate stochastic discount factors, therefore tend to be larger compared to dividends given that even far-away payments are highly valued. The resulting dividend yields, F_t, F_t^*, which divide dividends at market values, $p_t Y_t, p_t^* Y_t^*$, by equity prices, Q_t, Q_t^*, are therefore significantly smaller on average (Figure F.30). Because the impact of the tax on foreign dividends is ultimately governed by the magnitudes of the dividend yields given that the differences in equity premia as perceived

[63] This switch in sign can be observed by comparing Figures F.25 and F.28 in Appendix F. Figure F.4 shows the corresponding distribution, obtained as before by simulating the economy for 250 years. Accordingly, the relationship between fundamentals y_t and the wealth share x_t is now positive. This switch can of course have long-term consequences in terms of the surviving agent in the very long run: e.g. if the domestic good becomes dominant in the long run, the domestic investor will tend to dominate the economy under modest degrees of imperfect financial integration, while the foreign investor would have dominated under perfect risk sharing. Imperfect financial integration also have an effect on the dispersion of the wealth share around its broad relationship with y_t: as τ increases, x_t moves further away, underlying the fact that when financial markets are imperfectly integrated, international investors have a more difficult time sharing risk with each other. This is the result of the effect of market imperfection on risk premia and portfolios, described above, but also on consumption, driven by J_t, and taxes themselves. Figures F.24 and F.26 in Appendix show $\mu_{x,t} x_t$ and its decomposition into its several components, for $\psi = 0.2$ and $\tau = 10\%$.

by the two investors are $-\tau^* F_t, -\tau F_t^*$, their quantitative ability to impact risk premia, and therefore portfolios, is therefore much more limited when ψ is large. Economically, this is also related to the fact that the extent of bond trading, which becomes important when risk sharing is imperfect, becomes much more limited as ψ increases as seen in Panel (b) of Figure 10. Interestingly, the tax on foreign dividends has a limited effect in that case, even though the diversification benefits provided by the two equity assets appear smaller with an average correlation of returns of 0.91 against 0.77 with $\psi = 0.2$. This reflects the fact that looking at average correlations might not be an accurate enough measure of diversification benefits in contexts in which the correlation is inherently state-dependent like here.

Figure 10: Share of bond in the domestic portfolio

(a) $\psi = 0.2$ (b) $\psi = 2$



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except for ψ and $\mu = 10\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

The impact of imperfect financial integration also depends on the home bias in consumption and risk aversion that both increase the impact of the wealth share, the former by making investors more heterogeneous and therefore less able to share risk, and the latter by increasing the impact of the effect on investors' decisions. Like the main mechanism above, this also occurs when ψ is high, but is significantly more muted. Overall, those pieces of evidence point to the significant role played by the several dimensions of preferences in modulating the effect of the wealth share on the equilibrium, and therefore the effect of imperfect financial integration, a fact that could not have been studied so far in the literature given that it focused for the most part on special cases.

Overall, imperfectly integrated markets have a profound impact on the equilibrium, which is intimately related to the rising influence of the wealth share. This is consistent with investors no longer being able to share risk perfectly when there are frictions in market integration, so that the identity of the investor holding most assets in equilibrium, captured by the wealth share, matters more. Taken together, those results emphasize the intricate interplay between portfolio choices, asset prices, and risk sharing in this context, and imperfect financial integration has the potential to strikingly change the portfolios of international investors. To be sure, matching portfolios throughout the state space is no easy fit, and as we have seen, portfolios remain strongly state-dependent, in fact even more so than in the specification with perfect risk sharing studied thus far. In addition, whether imperfect financial integration has a strong enough impact to overturn the foreign bias in equity holdings that obtains in the baseline (provided that goods are good enough substitutes) depends significantly on the calibration of preferences, a fact that we have been able to uncover thanks to the generality of the framework. Yet, with those caveats in mind, imperfect financial integration of the form studied in this section remains a realistic and plausible way to generate a home bias in equity holdings broadly in line with the data – both qualitatively, and quantitatively for a relevant part of the state space. It is therefore adequate for our purposes, and I focus on this specification for the application of Section 4.

Investor heterogeneity The heterogeneity of investors is another factor that strongly reinforces the influence of the wealth share on the equilibrium, not only conceptually but also quantitatively.

This was already apparent in the analysis of the baseline calibration studied so far in which an increase in the home bias in consumption, which constitutes the fundamental heterogeneity in the economy, increases the impact of the wealth share significantly. This is true of both the direct effect of the wealth share as a state variable capturing the average investor in the world economy, and of the hedging of wealth risk, which becomes as important a determinant of portfolios as other components (Table 1).

Labor income is another way to introduce heterogeneity in the framework, while remaining in a symmetric calibration. This happens because labor income, modeled here as a constant share (δ, δ^*) of the output of each tree being paid to the local investor, makes the budget constraint of each investor more dependent on local conditions. This analysis is relegated to Appendix A.8 in the interest of space but labor income has a strong impact on the equilibrium and its underpinnings. While its effect on risk premia and Sharpe ratios is somewhat modest, it significantly affects portfolios, marginal values of wealth, consumptions, and the interest rate.[64] Most importantly, and in line with the emerging theme of this section, this effect goes hand-in-hand with a bolstered importance for the wealth share. As a stark example, the share of portfolio variance explained by the hedging of x_t increases from

[64]For portfolios specifically, labor income reinforces the bias in portfolio holdings. This comes from the fact that labor income is perfectly correlated with the payoff of the local asset, so that it renders each asset yet more attractive/unattractive to the local investor depending on the elasticity of substitution across goods. In the baseline for instance, labor income reinforces the foreign bias in equity holdings on average so that "The International Diversification Puzzle Is Worse Than You Think" (Baxter and Jermann, 1997). Importantly, this effect is also strongly state-dependent, and in particular relevant as the wealth share of an investor gets small.

7% in the baseline without labor income, to a whooping 70% for $\delta = 62.5\%$, a calibration roughly in line with the average labor share in the United States over the last 50 years. On the contrary, the common and y_t-hedging components now explain a mere 20% and 10%, instead of 70% and 23% in the baseline. In short: the hedging of wealth share risk becomes the main driver of the shape of portfolios. More generally, the direct of the wealth share as a state variable is also greatly reinforced.[65]

In summary, the heterogeneity of investors therefore makes the wealth share an important variable of interest in this framework. This aspect will also be particularly apparent when we turn to the application of Section 4 in which a different kind of heterogeneity, in the form asymmetries in preferences, takes center stage.

Taking stock, the characterization of asset prices and global portfolios emphasizes the importance of the allocation of wealth in this general international economy. The allocation of wealth matters both as a state variable that captures the average international investor, and as a pricing factor against which investors hedge. The magnitude of the impact of the allocation of wealth can grow substantially with imperfect financial integration, and when investors become more heterogeneous. In other words, "capital is back" in this international context too: consistent with a broader emerging literature in economics, the allocation of capital, here across international investors, has a prime role in determining economic outcomes.

In terms of portfolios, home bias in equity holdings like in the data can obtain in the setup either when the elasticity of substitution across goods is low, or due to imperfect financial integration provided that the elasticity of intertemporal substitution is moderate. In what follows, I focus on the latter case, which is both realistic – international markets are likely to be imperfectly integrated in practice –, and because it is more consistent with standard estimations of the elasticity of substitution across goods. As we have seen in that case, the home bias is amplified by the hedging of the wealth share risk. More generally, portfolios as well as other variables strongly vary throughout the state space, emphasizing the importance of the global solution. All those aspects are present and even reinforced in the application of the model to which I now turn.

[65]The way those patterns change when considering a more general and realistic specification for labor income could prove an interesting exploration. One particular specification could be to construct labor income as a time-varying share of the output of each country, as explored for instance in Coeurdacier and Gourinchas (2016). As the authors suggest, the correlation of labor income with output, once computed with the proper conditioning, could in fact turn out to be negative, providing a natural way to generate a home bias in equity holdings. If the share is itself stochastic, it could also provide an additional hedging motive that could prove relevant in practice also as it introduces a natural degree of market incompleteness. Labor income could also take a more general form, for instance as a separate source of idiosyncratic risk in the spirit of the recent heterogeneous-agent macroeconomic literature like Kaplan et al. (2018), or by introducing a distribution of investors in each country by generalizing the overlapping generation structure of Gârleanu and Panageas (2015) to a two-good, two-country setting. I leave these promising avenues for future research.

4. Application: The International Financial System

Because of its generality, the framework in this paper represents a versatile building block towards several applications and extensions. In this section, I specialize the model to capture important dimensions of the international financial system, with a particular focus on the role of the United States, its center country. The introduction of asymmetries in the tolerance for risk of international investors naturally replicates the role of the United States as the world banker, documented in Gourinchas and Rey (2007b) and Gourinchas et al. (2017), and a modest degree of imperfect financial integration also generates a plausible home bias in equity holdings for both investors. Importantly, those additions not only allow us to match facts on the U.S. external portfolio on average, but also make it possible to study its dynamics. In particular, crisis episodes, by worsening the wealth position of the world banker, lead to a sharp increase in global risk aversion that in turn increases risk premia worldwide, in a pattern reminiscent of some aspects of the Global Financial Cycle proposed by Rey (2013) and Miranda-Agrippino and Rey (2020) but for which a general equilibrium exploration remained elusive. The framework allows to study the response of portfolios to shocks, the process of external adjustment of the center country, as well as the evolution of the (time-varying) comovement of returns. In doing so, it is able to replicate several additional facts e.g. about asset return dynamics. I also discuss a number of counterfactual experiments, as well as worthwhile extensions that pertain to the introduction of global financial intermediaries in this international context. Overall, those findings reinforce the broad message that emerged from the characterization of Section 3: capital is back, and the allocation of wealth is of prime importance in this context.

4.1. Stylized facts

I start by summarizing a number of stylized facts about the international financial system and asset returns in this context, which the specialization of the model can jointly match. I focus in particular on the role of the United States, its center country.

Fact 1 (U.S. as world banker)**.** *The United States borrows from the rest of the world in safe assets, and uses it to lever up its investment in risky assets worldwide, resulting in a negative net foreign asset position. The country therefore plays the role of the world banker. (Gourinchas and Rey, 2007b; Gourinchas et al., 2017)*

Fact 2 (Exorbitant privilege)**.** *The United States earn excess returns on average on its net foreign asset position, in particular in normal (non-crisis) times. (Gourinchas and Rey, 2007b; Gourinchas et al., 2017)*

Fact 3 (Exorbitant duty)**.** *In times of crisis, the United States plays the role of the world insurer, transferring wealth to the rest of the world. This exorbitant duty is the flip of its* *exorbitant privilege in normal times, and is associated with a strong deterioration in the U.S. net foreign asset position. (Gourinchas et al., 2017)*

Fact 4 (Home bias in equity holdings)**.** *The aggregate portfolio of the United States, as well as that of most countries around the world, exhibit a strong bias towards domestic equity securities. This bias has slowly decreased in recent decades. (French and Poterba, 1991; Coeurdacier and Rey, 2013)*

Fact 5 (Global Financial Cycle)**.** *Periods of global stress are characterized by risk-off scenarios in which the global risk aversion, as well as risk premia worldwide, spike up. (Some aspects of Rey, 2013; Miranda-Agrippino and Rey, 2020)*

Fact 6 (International Financial Adjustment)**.** *Periods of strong deterioration in the net foreign asset position of the United States predict higher expected risk premia on its external balance sheet in the short to medium term, consistent with valuation effects playing a key role in the process of its external adjustment. (Gourinchas and Rey, 2007a, extended to more recent data in Gourinchas et al., 2019)*

Fact 7 (Countercyclicality of asset return dynamics)**.** *The (i) risk premia, (ii) Sharpe ratios, (iii) volatilities, and (iv) correlation of risky returns are countercyclical, i.e. they increase in times of crisis. This is true in particular in times of global stress. Those evolutions are consistent with crises being periods in which not only the quantity of risk rises, but also the price of risk that is received as a compensation. (Among others, in the context of the United States: for (i), (ii), (iii), cf. Lettau and Ludvigson, 2010; for (i), cf. Fama and French, 1989, Ferson and Harvey, 1991, Harrison and Zhang, 1999, Campbell and Diebold, 2009; for (ii), cf. Harvey, 2001; for (iii), cf. Schwert, 1989, Brandt and Kang, 2004)*

Fact 8 (Global trend in asset comovements)**.** *The comovement of equity prices worldwide has increased over the last 150 years, and particularly rapidly in the past three decades. This increase goes above and beyond the growing synchronization in real sector variables. The sharp increase in the comovement of global equity markets is driven by fluctuations in risk premia, which are themselves strongly impacted by fluctuations in global risk appetite. (Jordà et al., 2019)*

Fact 9 (Global trend in interest rates)**.** *The real rate of interest has trended down globally in recent decades, and this can be related to the relative size of the risk-tolerant world banker (the United States) decreasing in the world economy. (Caballero et al., 2008; Hall, 2016; Gourinchas et al., 2017)*

4.2. External portfolio and the exorbitant privilege

Throughout Section 4, the domestic country is taken to represent the United States, the country at the center of the international financial system. Its representative investor is assumed to display a larger appetite for risk, so that her risk aversion is now lower than that for the representative foreign investor ($\gamma = 8 < \gamma^* = 15$). This assumption, in the spirit of Caballero et al. (2008), Gourinchas et al. (2017), and Maggiori (2017), is meant to capture the greater development and depth of U.S. financial markets, making the country as a whole better able and willing to carry financial risk in the world economy. Following the discussion in Section 3.5, I also introduce a modest degree of imperfect financial integration, $\tau = \tau^* = 15\%$, which, combined with an elasticity of intertemporal substitution of $\psi = \psi^* = 0.5$, generates plausible portfolios, as well as a realistic level for the interest rate. Other parameters are calibrated as before and summarized in Assumption 1.

The introduction of asymmetries in the tolerance for risk of the two international investors has a profound impact on the equilibrium and in particular on countries' portfolios. Specifically, the representative U.S. investor is willing to borrow from risk-averse investors in the rest of the world using the risk-free bond, so as to lever up her risky portfolio. Panel (a) of Figure 14 shows that this borrowing is large, and happens throughout the state-space. For instance, around $x_t = 30\%$, which corresponds broadly to the share of the United States in world wealth (**?**), the country borrows about 50% ($b_t = -0.5$) of its wealth in international markets, so as to invest 150% of its wealth in risky equity securities. Those results are consistent with the findings of Gourinchas et al. (2017), who document a strongly negative net foreign position in safe securities for the United States, which uses those safe liabilities to finance its investments in risky assets worldwide.[66] In other words, the model naturally replicates Fact 1 about the role of the United States as the world banker, or more accurately, given the amount of leverage, as the world venture capitalist, as pointed out in Gourinchas and Rey (2007b). As a result, the net foreign asset position of the United States is strongly negative, like in the data. In the model, the latter is computed as the difference between the wealth invested in the foreign equity asset and in the world bond by the domestic investor, and that invested in the domestic asset by the foreign investor. As a fraction of domestic wealth, this yields

$$\frac{NFA_t}{W_t} = w_{f,t} + b_t - w^*_{h,t}\left(\frac{1-x_t}{x_t}\right) \tag{21}$$

The results is shown in Panel (b) of Figure 14. At $x_t = 30\%$, $NFA_t/W_t = -30\%$ on average. Like the amount of borrowing, the net foreign asset position also varies strongly with the state of the economy as a result of both asymmetries and imperfect risk sharing, an aspect that I discuss when focusing on dynamics below.[67]

[66] I should note that even though the framework gets us close to several empirical measures, such as the net foreign asset position, interest rates, or portfolios, the goal of the exercise is primarily conceptual. A number of small additional extensions could be considered to make the quantitative match even closer.

[67] The international bond can also be considered to be part of the domestic country foreign *liabilities*,

The modest degree of imperfect financial integration is able to generate a plausible home bias in equity holdings for both investors (Fact 4). With $\tau = 15\%$, the shares in the domestic equity portfolio allocated to domestic and foreign assets for $x_t = 30\%$ are around 70% and 30% respectively, broadly in line with findings in Coeurdacier and Rey (2013) for recent years.[68] This result is of course subject to the caveats discussed in Section 3.5. First, as I discuss below and as seen in Figure 13, portfolios again vary substantially with the state of the economy so that matching them in one given point does not guarantee matching them throughout. This is particularly true here where the impact of the allocation of wealth on portfolios is significantly larger than in the baseline of Section 3. Second, for tax on foreign dividends to have a quantitatively sufficient effect, the elasticity of intertemporal substitution ψ must not be too large, and I pick a value of $\psi = 0.5$ to also be able to match the average level of the interest rate.[69] Despite those caveats, the degree of imperfect financial integration allows to generate portfolios that are broadly consistent with the data and is therefore adequate for our purposes.

Finally, as a last point in terms of average levels, the model is also able to reproduce the exorbitant privilege that the United States has benefited from as a result of its external portfolio (Fact 2). I follow the definition of Gourinchas et al. (2017), and understand exorbitant privilege to mean the excess returns on the U.S. external portfolio. This is first visible by comparing the expected returns on the *total* portfolio of the domestic investor, $\overline{\mu_{R,t}} \equiv w_{h,t}\mu_{R,t} + w_{f,t}\mu_{R^*,t} + b_t r_t$, to that of the foreign investor, $\overline{\mu_{R^*,t}} \equiv w^*_{h,t}\mu_{R,t} + w^*_{f,t}\mu_{R^*,t} + b^*_t r_t$. Panels (a) and (b) of Figure 11 plot both as a function of the wealth share. The former is larger regardless of the state of the economy, with $\overline{\mu_{R,t}} = 4.9\%$ and $\overline{\mu_{R^*,t}} = 3.9\%$ on average, reflecting the riskier position taken by the United States that is financed by borrowing internationally in the safe asset and earns a higher returns in expectation. Interestingly, this result highlights the fact the center country borrows to invest not only internationally but in its own domestic assets as well.

Second, we can also focus specifically on the returns on the *external* portfolio itself. Here, the United States earns $r^a_t = \mu_{R^*,t}$ on its external assets, which are comprised of the foreign equity asset, and pays on its external liabilities r^l_t, the weighted average of the returns on the domestic equity and on the risk-free bond. The excess returns on the external position

reflecting the fact that most countries save internationally using Treasury bonds issued by the United States. This does not change the accounting equation given that one would subtract $-b^*_t(1-x_t)/x_t$, which is equal to b_t by market clearing. (Recall that $b_t < 0$ as the United States borrows from the rest of the world.) The measure can also be expressed as a fraction of domestic output as follows

$$\frac{NFA_t}{Y_t} = \left(w_{f,t} + b_t - w^*_{h,t}\left(\frac{1-x_t}{x_t}\right)\right)\left(\frac{x_t}{z_t}\right)\left(\frac{(1-\delta)p_t}{F_t}\right) \tag{22}$$

[68] The corresponding measure, $1 - FB_t$, is of 40% on average for $x_t = 30\%$, slightly below empirical estimates, but varies substantially throughout the state space.

[69] An additional source of asymmetry that could be worthwhile to study in this context is differences in ψ. This could capture differences across countries in propensity to save, for instance between the United States, and heavy savers like Europe due to its demographic deficit, and China or southeast Asia.

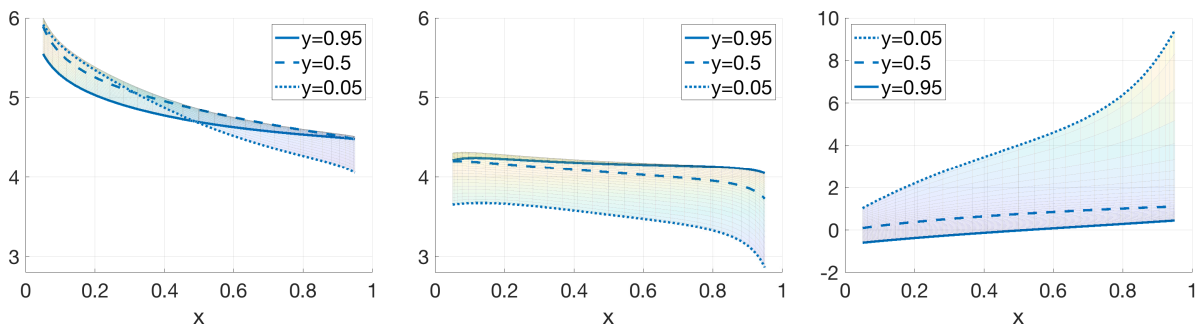
are therefore

$$r_t^a - r_t^l \equiv \mu_{R^*,t} - \left\{ \left(\frac{w_{h,t}^*}{w_{h,t}^* + b_t^*} \right) \mu_{R,t} + \left(\frac{b_t^*}{w_{h,t}^* + b_t^*} \right) r_t \right\} \tag{23}$$

Panel (c) of Figure 11 shows the results, with $r_t^a - r_t^l = 1.4\%$ on average, and about 0.7% when $x_t = 30\%$, consistent with the lower range of estimates in Gourinchas et al. (2017).[70] The excess returns earned on its external portfolio allows the United States to finance its external deficit, and helps reduce the burden of its external adjustment process, as discussed in Gourinchas and Rey (2007a) and below.

Figure 11: Average expected returns on countries' portfolios (%)

(a) United States: $\overline{\mu_{R,t}}$ (b) Rest of the world: $\overline{\mu_{R^*,t}}$ (c) United States: $r_t^a - r_t^l$



Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

In summary, the introduction of asymmetries in risk aversion, together with a modest degree of imperfect financial integration, is able to reproduce a number of stylized facts about the average level of the external portfolio of the United States in the world economy. Specifically, those additions allow the framework to capture Facts 1, 2 and 4 of Section 4.1

[70] In this framework, we cannot easily replicate the fact the United States not only earns a higher returns on average, but also within asset categories. Indeed, because of the home bias in equity holdings that obtains with imperfect financial integration, the returns on the foreign equity asset are in fact larger than that on the domestic equity asset for a large part of the state space, due to the fact that the foreign asset is held for a larger proportion by the foreign investor who is more risk-averse and therefore requires a higher compensation for risk on average. This is not true for the whole state space however, and one could potentially obtain a larger $\mu_{R,t}$ by reducing the impact of the home bias in equity holdings. This could also be alleviated for instance by introducing more assets, in the form of other countries/trees or several bonds. Those explorations are left for future research.

4.3. Crisis, the Global Financial Cycle, and dynamic aspects

The better ability and willingness of the domestic country to bear financial risk not only naturally replicates the external portfolio of United States in levels, but also can help jointly match a number of facts about the dynamics of the international financial system and asset returns.

To see this, let us consider a negative shock to the output of the United States, the domestic country ($dZ_t < 0$). This shock, because it leads the domestic asset to do particularly poorly, brings a sharp decline in the share of wealth held by the domestic country, due to the home bias in equity holdings that results from imperfect financial integration on average. Because it affects the country most able to carry risk in the world economy, it is meant to capture a severe international crisis of the type the world experienced in the Great Recession of 2008.[71]

The decrease in the share of world wealth held by the domestic country directly captures the transfer of wealth that occurs from the United States to the rest of the world in times of international crisis. As put forward by Gourinchas et al. (2017) and summarized in Fact 3, this is the flip side of the banker role played by the United States in international markets. In good times, the world banker reaps the exorbitant privilege of its role by earning higher returns and running a large negative net foreign asset position, as we have seen. In bad times however, the country bears the exorbitant duty of insuring the rest of the world by transferring wealth to other countries. Empirically, consistent with my results, the exorbitant duty is large: Gourinchas et al. (2017) estimate that the transfer of wealth amounted to around 19% of U.S. GDP during the 2007-2009 global financial crisis. In the model, the phenomenon can also be observed clearly from the patterns of consumption, with the share of consumption (at market value) enjoyed by the United States declining monotonically with its share of world wealth, consistent with the rest of the world receiving a transfer and therefore consuming more in times of crisis.[72]

[71] The loadings of the wealth share on both shocks is shown in Figure F.20. A negative shock to the foreign output ($dZ_t^* < 0$) also leads to a decrease in the share of wealth of the domestic country in most of the state space because even though its representative investor holds more of her wealth in the domestic asset, she still holds a larger share of the foreign asset than the foreign investor. This is because she levers up her risky portfolio using the international bond. However, a shock to domestic output has a stronger effect due to the home bias in equity holdings, and is more akin to a type of world shock like the Great Recession of 2008 or the Global Pandemic of 2020. The processes of output could also be specified so as to be driven by a common component, for instance by adapting the share process of Santos and Veronesi (2006) to capture the evolution of relative supply y_t, as briefly discussed in Section 4.5. I leave this exploration for future research, but note that it could also help deliver a stationary distribution of world wealth.

[72] The share of domestic consumption at market value is computed as

$$\frac{P_t C_t}{P_t C_t + P_t^* C_t^*} = \frac{c_t x_t}{c_t x_t + \mathcal{E}_t c_t^* (1 - x_t)} \tag{24}$$

In addition and most interestingly, the global shock also gives rise to movements in asset premia that are reminiscent of some aspects of the Global Financial Cycle put to light by Rey (2013) and Miranda-Agrippino and Rey (2020), but which have so far not been captured in a general equilibrium model of the international financial system. As the share of wealth held by the United States decreases, global wealth-weighted risk aversion γ_t, which is time-varying and state-dependent in this asymmetric context, spikes up, reflecting the fact that the rest of world, financially less able and willing to carry risk, governs a larger share of world wealth following the wealth transfer.[73] This pattern captures the emergence of a risk-off scenario worldwide with the compensation for taking risk rising globally as the wealth position of the world banker deteriorates, and is one of the markers of the Global Financial Cycle summarized in Fact 5. In turn, it leads asset prices to decrease more than they would due to the sole effect of worse fundamentals, so that the risk premia and Sharpe ratios on both assets increase sharply. Similarly, the interest rate declines globally with the world becoming more risk-averse on average. This captures the global risk factor of Miranda-Agrippino and Rey (2020) and the second element of Fact 5. Figure 12 shows each of those variables as a function of the wealth share, and suggests that the impact is large. Compared to the baseline of Section 3 in which the wealth share had a modest effect on risk premia, it now becomes a much more important determinant, on par with fundamentals captured by the relative supply of goods y_t. For instance, the interest rate decreases by 73 basis points when the domestic country goes from being dominant to small in world wealth, and risk premia increase by 25 to 60 basis points depending on the asset. Those effects are large given that the overall level of risk premia remains modest in this economy in which the only friction is a mild degree of imperfect financial integration. For instance, they represent increases in risk premia of 22 to 51%. The effect is also clear when risk premia are normalized by units of volatility: Sharpe ratios increase by 0.12 to 0.17, a 33 and 49% increase respectively. Quantitatively, this reinforced effect can also confirmed by computing elasticities with respect to each state variable.[74] The main economic message is that, in this economy that features international investors with diverging tolerance for bearing risk, the identity of the ultimate holder of most assets in the world economy, captures by x_t, matters as much or more than the fundamental underpinnings of those assets, captured by the relative supply y_t. In other words, once again, "capital is back": the allocation of wealth has a profound impact on the world economy.

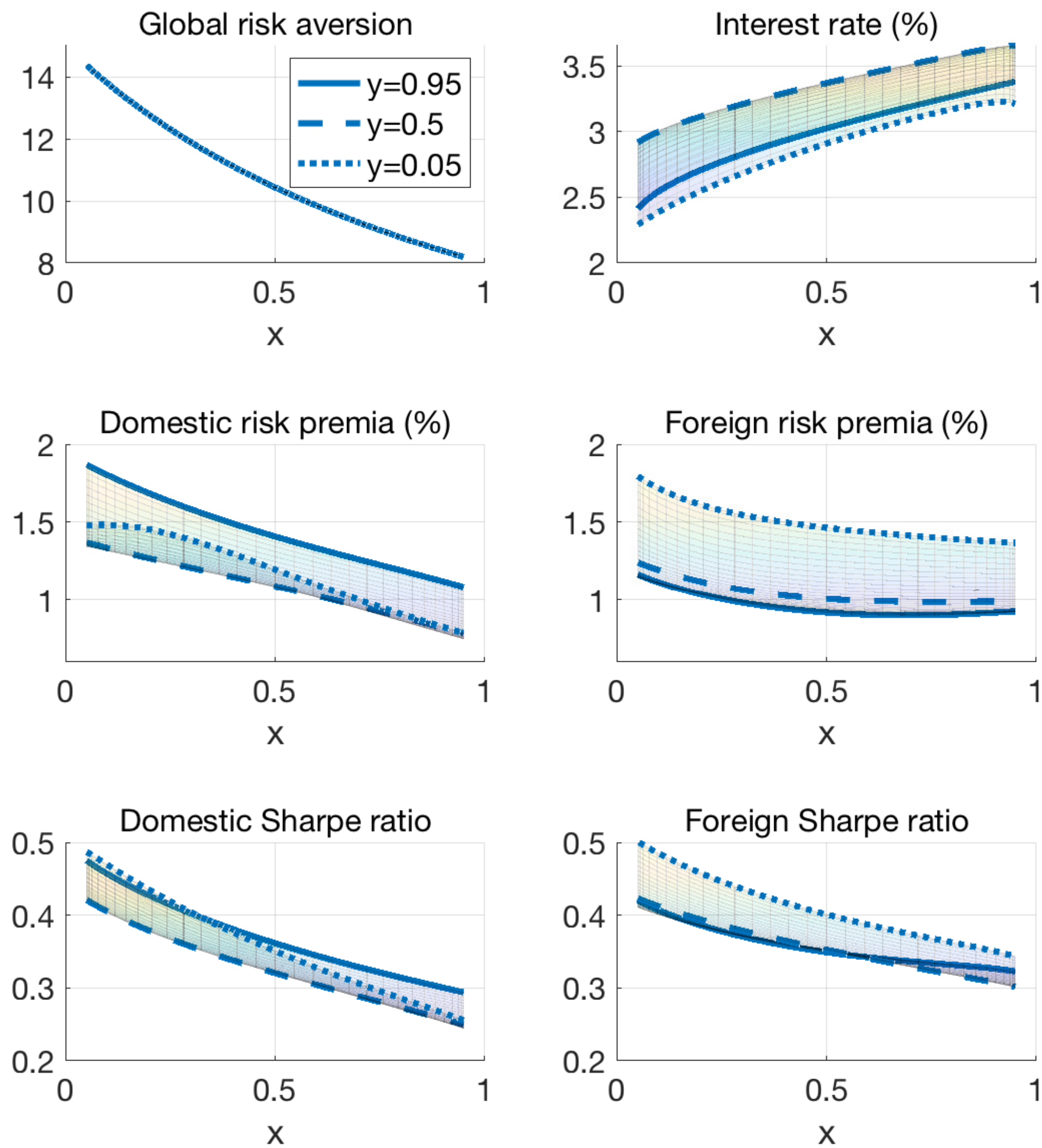
To be sure, the mechanism proposed here does not cover all dimensions of the phenomenon documented by Rey (2013) and Miranda-Agrippino and Rey (2020). For one, because the model is real, there is not role for monetary policy. In addition, the type of risk-off scenario

[73]As a reminder, the global wealth-weighted risk aversion is defined as $\gamma_t \equiv \left(\frac{x_t}{\gamma} + \frac{1-x_t}{\gamma^*}\right)^{-1}$.

[74]Specifically, we can compute $|\varepsilon_{g_t,x_t}| \equiv \left|\frac{\partial \ln g_t}{\partial \ln x_t}\right|$ and $|\varepsilon_{g_t,y_t}| \equiv \left|\frac{\partial \ln g_t}{\partial \ln y_t}\right|$ where $g_t \in \{\mu_{R,t} - r_t, SR_t\}$. On average, $\overline{|\varepsilon_{\mu_{R,t}-r_t,x_t}|} = 0.41$ and $\overline{|\varepsilon_{SR_t,x_t}|} = 0.32$ on par with $\overline{|\varepsilon_{\mu_{R,t}-r_t,y_t}|} = 0.59$ and $\overline{|\varepsilon_{SR_t,y_t}|} = 0.35$. This is much larger than in the baseline in which $\overline{|\varepsilon_{\mu_{R,t}-r_t,x_t}|} = 0.02$, $\overline{|\varepsilon_{SR_t,x_t}|} = 0.03$, $\overline{|\varepsilon_{\mu_{R,t}-r_t,y_t}|} = 0.54$, $\overline{|\varepsilon_{SR_t,y_t}|} = 0.22$. The effect is present in particular in relevant parts of the state space. For instance, when $x_t = 30\%$, average elasticities are $\overline{|\varepsilon_{\mu_{R,t}-r_t,x_t}|} = 0.03$, $\overline{|\varepsilon_{SR_t,x_t}|} = 0.04$, on par or above the effect of fundamentals, $\overline{|\varepsilon_{\mu_{R,t}-r_t,y_t}|} = 0.01$, $\overline{|\varepsilon_{SR_t,y_t}|} = 0.05$.

that arises at the international level could also be taking place within countries, with a shift towards safe assets for the average investor of each country. I discuss this point in the context of the reserve currency paradox below. Moreover, because there is only one international bond, which is riskless and pays the same interest for all investors, it does not capture the type of convenience yields that has been discussed as an important element in this context, e.g. by Jiang et al. (2020), and Kekre and Lenel (2020).

Figure 12: Asset pricing under asymmetric risk tolerance and imperfect financial integration



Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Another relevant aspect can also be to impose portfolio constraints so that the investors of both countries, but in particular that representative of the United States, have time-varying limited risk-bearing capacity that vary with their balance sheets. Because the framework is flexible, those dimensions can be introduced and represent promising avenues for future research. I discuss some of them in Section 4.5. Still, the main economic message remains: the introduction of asymmetries in the capacity and tolerance to bear risk together with a mild degree of imperfect financial integration, which naturally replicate the aggregate portfolio of the United States on average, also give rise to dynamic aspects that had so far not been clearly elucidated and that are reminiscent of important dimensions of the evolution of the international financial system in times of severe global crisis.

Importantly, because I study this question in a general international portfolio choice context, the framework also allows to discuss and match a number of further relevant facts.

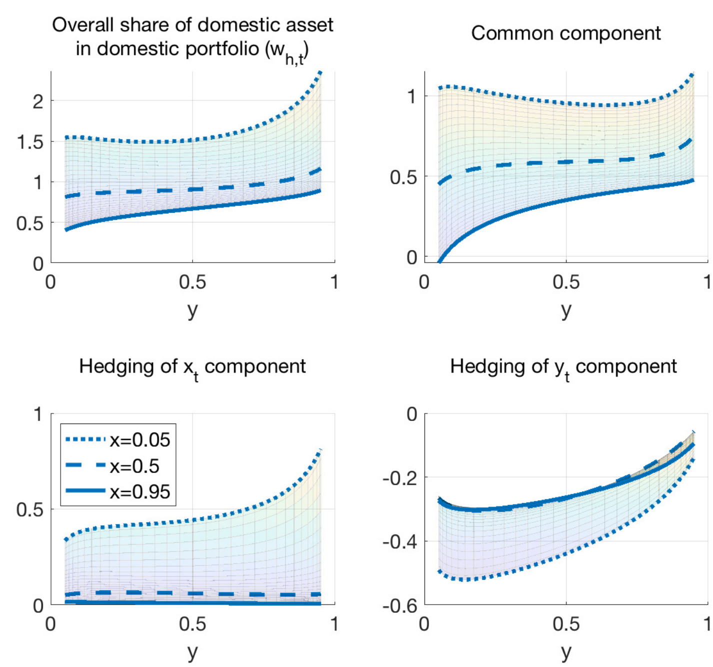
First, portfolios can be analyzed not only on average, but also in their dynamics. In particular, even though the introduction of imperfect financial integration allows the model to match the home bias in equity holdings qualitatively, and quantitatively around $x_t = 30\%$, portfolios remain strongly state-dependent. As her wealth share decreases following the shock, the representative domestic investor tends to increase the weight of risky assets further in her portfolio, by ramping up her borrowing from the rest of the world as a share of her wealth. Her desire to do so comes both from rising risk premia worldwide in particular on the domestic asset, which impact her portfolio via its common component, and from the hedging of her wealth share risk. The latter is particularly active throughout the state space and it strongly reinforces the home bias in portfolio holdings due to the positive equilibrium relationship between wealth share and relative supply that obtains in this case, which leads the domestic asset to pay increasingly well as the domestic marginal value of wealth increases in the wealth share dimension (cf. Section 3.5 for details). This effect is especially strong as the wealth share of the domestic investor becomes small as seen in Figure 13. Quantitatively, the impact of the wealth share is once again significantly larger than in the baseline of Section 3, due to the combination of asymmetries in preferences and imperfect financial integration. This is true both for its direct effect as a state variable that captures the average investor in the world economy – the two international agents are now very different–, and through its hedging effect – a variance decomposition similar to Section 3.4 suggests that the hedging of x_t now drives 39% of the changes in $w_{h,t}$ against 7% in the baseline.[75]

Portfolios also vary strongly with the relative supply of goods, due to effect of the home bias in consumption combined with asymmetries in risk tolerance and imperfectly integrated markets. As y_t decreases, which happens when the negative shock is concentrated on the domestic output, the domestic investor chooses to increase her share in the foreign asset, $w_{f,t}$, so that it locally can rise above that of the domestic asset, $w_{h,t}$. This reflects for a large

[75]For $w_{h,t}$, in more details, the common, hedging of x_t, and hedging of y_t components drive 69%, 39%, and -2%, respectively. For $w_{f,t}$, the contribution of the common, hedging of x_t, and hedging of y_t components are of 19%, 17%, and 37%, with the tax component accounting for the remaining 27%. In the baseline of Section 3, the first three figures were 70%, 7%, and 34% respectively, for both $w_{h,t}$ and $w_{f,t}$.

part the common component of portfolios, with the share of the domestic asset in the market portfolio, z_t, being particularly small in that case, as well as the pattern of bond trading. Indeed, while the fact that the domestic investor borrows on average is driven by her higher tolerance for risk, the trading in the bond is also asymmetric: she borrows in particular as y_t decreases, driven by the introduction of imperfect risk sharing, and as discussed in Section 3.5. When compared to the weight in the market portfolio z_t, the HB_t measure remains above 1, and in fact much so in this particular region of the state space. As the relative supply of the domestic good y_t increases however, which would happen if the foreign output is especially negatively impacted, the domestic investor borrows a lesser share of her wealth, and uses it to lever up mostly on the domestic asset whose expected returns increase particularly much, while she barely invests in the foreign asset, $w_{h,t} \gg w_{f,t}$. In this case, the weight on the foreign asset is not only small in itself, but also when compared to its weight in the market portfolio, $1 - z_t$.

Figure 13: Components of the domestic portfolio



Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Taken together, those results highlights once again that portfolios are strongly time-varying in this context, in fact more so than in the baseline case of Section 3. Once again, assessing whether those patterns are borne out in the data calls for additional international portfolios facts. From that perspective, the large and detailed data gathering effort undertaken for the Global Capital Allocation project of Maggiori et al. (2020) and Coppola et al. (2020) will assuredly prove invaluable. Especially important would also be the establishment of facts about portfolio rebalancing at a higher frequency and in response to shocks. If portfolios instead do not vary as much, this could be the sign of a failure to optimize in the part of international investors, in particular in times in which it is most valuable.

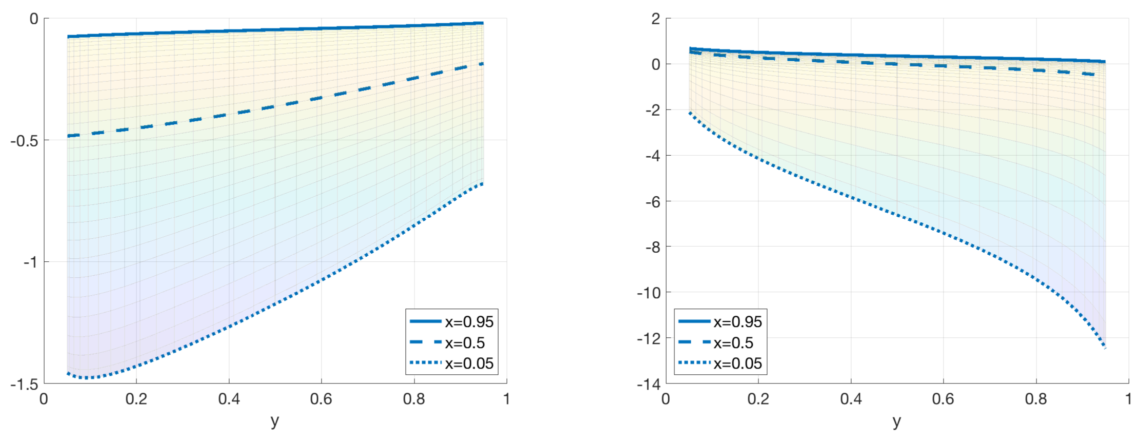
Second, the process of external adjustment of the country at the center of the international financial system can also be studied in this context. As the shock hits, not only does the share of world wealth held by the United States decrease, reflecting the wealth transfer to the rest of the world coming from its role as world insurer, but the net foreign asset position of the country also strongly deteriorates. This is shown in Panel (b) of Figure 14, and is consistent once again with the findings of Gourinchas et al. (2017) in Fact 3. In this context, the phenomenon arises because the United States borrows an increasing share of its wealth as its wealth share decreases, as shown in Panel (a), broadly in line with the increased burden of its safe liabilities to the rest of the world that results empirically in a sizable decline of the net foreign safe asset of the country further into negative territory in times of crisis. In addition, the sharp increase in expected risk premia that occurs simultaneously emphasizes the primordial role in this situation of valuation effects of the type proposed by Gourinchas and Rey (2007a) for the process of external adjustment. In my model, they take the form of time-varying risk premia: the United States invests heavily in risky assets whose expected returns are large in particular in times of crisis. Those ensures that its net foreign asset position is expected to improve in the future, thereby facilitating the process of external adjustment and allowing the country to sustain an even more negative external position when the crisis hits. In the long run, the higher returns earned on risky assets on average lead the domestic country to accumulate wealth at a faster rate so that the domestic investor progressively dominates the world economy. This is accompanied by a positive net foreign asset position, which also participates in easing the pressure of the external adjustment process even when the net foreign asset position is negative at shorter-term horizons.[76] I come back to long-run dynamics briefly in Section 4.4. Accordingly, the combination of the dynamics of the net foreign asset position of the domestic country and that of expected returns also replicates Fact 6 about the predictability regressions of Gourinchas and Rey (2007a), and extended to more recent data in Gourinchas et al. (2019). In the short run, a deterioration of the net foreign asset position of the United States is indeed associated with higher expected returns, $\mu_{R,t} - r_t, \mu_{R^*,t} - r_t$, as documented previously when the shock hits. This pattern persists for medium-term horizons with risk premia remaining elevated for the time that the net foreign asset position stays below its initial level. Yet, as the wealth share and net foreign asset position continue improving back, owning to higher returns on the external portfolio on average, expected risk premia decrease back towards their initial level or below. The negative relationship between the initial change in net foreign asset position and expected risk premia therefore dampens with the horizon, as in the work of the authors.

[76]In fact, here, the net foreign asset position accumulated by the domestic country is such that the country can sustain a trade deficit even in the long run.

In the long run, x_t becomes increasingly large so that risk premia are reduced further due to the reduced global risk aversion, and the relationship between the initial change in net foreign asset position with those returns becomes muted like in the data.

Figure 14: External position of the domestic country

(a) Share of bond in domestic portfolio (b_t) (b) Net foreign asset position (NFA_t/W_t)



Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Finally, the framework also generates a number of asset pricing facts that are broadly in line with empirical observations, and that emerge here in an international context.

Firstly, the evolution of risk premia discussed previously shows that expected returns on both assets increase following a negative shock to the output of either of the country. This result is in line with countercyclical risk premia that have been documented in a wide variety of contexts (Fact 7, (i)), and emerges due to the heterogeneity between investors. In line with results in simpler one-good one-country contexts with stylized preferences e.g. in Weinbaum (2009), the share of wealth held by the risk-tolerant investor, here the investor representative of the domestic country, decreases with the shock, so that risk aversion and risk premia spike up, reflecting the fact that equity prices fall more than they would due purely to the worsening of fundamentals. Importantly, the sharp rise in risk aversion emphasizes that in this context risk premia are driven for an important part by large fluctuations in the compensation for risk, instead of solely being the result of changes in the quantity of risk. This is in line with a large literature that has seen changes in the price of risk emerge as a crucial explanation behind asset return predictability more generally.

In addition, in this general environment, the asymmetry in tolerance for risk interacts intimately with (i) the core heterogeneity that emerges due to the home bias in consumption, (ii) the fact that the two trees produce differentiated goods so that asset returns are impacted by changes in the prices of goods, and (iii) imperfect financial integration. For instance, for most values of y_t, the risk premium on the domestic asset increases faster than that on the foreign asset as the wealth share decreases. This reflects a larger change in ownership in this asset that the domestic investor tends to favor due to her home bias in equity holdings, which itself results from imperfect financial integration. As her wealth share decreases, the foreign investor has to pick up the slack, and the extra compensation for risk that the latter requires increases domestic returns comparatively more. Similarly, the effect of the relative supply on risk premia is also asymmetric. When the domestic country dominates the world economy, $\mu_{R,t} - r_t$ and $\mu_{R*,t} - r_t$ evolve broadly as expected and increase with the relative supply of the respective good underlying their payoffs. When the wealth share of the domestic country is small however, the risk premium on the domestic asset, $\mu_{R,t} - r_t$, also increases significantly when the relative supply of the *foreign* good is large, i.e. when y_t is small, reflecting in part the fact that its returns tend to locally comove negatively with the world-weighted marginal value of wealth (cf. Proposition 4). The patterns of risk premia also reflect the evolution of the dividend yields on the two assets, F_t, F_t^*, which are inherently state-dependent in this context (Figure F.21).

Conditional Sharpe ratios are also strongly countercyclical, increasing as the wealth share decreases following a shock to the output of any of the trees. This can be seen in Figure 12, and is consistent with results in Harvey (2001) and Lettau and Ludvigson (2010) for the United States (Fact 7, (ii)). This pattern is driven mostly by the evolution of risk premia, and happens despite the volatility of the two risky assets being also mildly countercyclical for a relevant part of the state space, as discussed below. In terms of levels, the Sharpe ratio on the domestic asset is smaller for most of the state space, reflecting the fact that the volatility of the domestic asset is larger on average, following the larger impact of the change in ownership mentioned above.

Turning to second moments more specifically, which are shown in Figure F.22, volatilities are slightly larger on average than in the baseline of Section 3 owning to the combination of additional heterogeneity and imperfect financial integration. They exhibit a mild countercyclicality, as observed empirically, in at least a relevant part of the state space (Fact 7, (iii))[77]. In particular, the volatilities of both assets, $\left(\sigma_{R,t}^T \sigma_{R,t}\right)^{-1/2}$, $\left(\sigma_{R*,t}^T \sigma_{R*,t}\right)^{-1/2}$ increase following the negative shock to U.S. output that has been the focus throughout the section. For the foreign asset, this occurs regardless of the wealth share of the domestic investor because the volatility of the foreign asset naturally increases following the decrease in y_t that also ensues. For the domestic asset, this occurs provided that the wealth share of the domestic country is not too small, and that the relative supply of the domestic good is not too large. Both are likely to happen following the shock considered here, and starting from a share of the United States in world wealth around $x_t = 30\%$ as in the data. More generally, Figure F.22 emphasizes that like their expectations, the second moments of returns are inherently time-varying, and non-monotonic. This is most visible beyond volatilities by looking at the conditional covariance and correlation of returns in Panels (e) and (f). Once

[77] This is the so-called "leverage" effect and has been the subject of a large literature, starting from Schwert (1989). Cf. also Brandt and Kang (2004), as well as Lettau and Ludvigson (2010) for an exploration of the patterns of both the first and second conditional moments of returns.

again, following the negative shock to domestic output, and as long as the effect is large enough on y_t as well, both can increase and are therefore countercyclical in line with Fact 7, (iv).[78] In this context, this happens because the covariance and correlation increase as the relative supply of the domestic good decreases, and as the wealth share of the domestic investor decreases, provided that the latter is not too large. Note that like for volatilities, both are strongly non-monotonic. For instance, the correlation tends to have a U-shape relationship with the wealth share: it decrease with the wealth share as x_t is small, but then starts *in*creasing with the wealth share, as soon as the x_t surpasses around 40 or 50%. This is in sharp contrast to the baseline in which the correlation of returns tended to reach its maximum around the symmetric point, $x_t = 50\%$ (Figure F.32).

Taken together, those results highlight that the deep and complex interactions between the several dimensions of the model – multiple goods, home bias in consumption, asymmetries, imperfect financial integration – yield asset pricing facts that are broadly consistent with empirical observations.

In summary, the introduction of asymmetric tolerance for risk in the framework studied in Section 3, combined with imperfectly integrated markets, allows to match not only the external portfolio of the United States, but also gives rise to a number of additional predictions that are strongly borne out in the data. Those results emphasize the value of being able to study those phenomena in a unified general equilibrium model of international portfolio choice. From a more theoretical perspective, here again, the impact of the wealth share not only on portfolios but also on asset prices, is strongly reinforced. This has shined through throughout the discussion, and is also reflected in the underlying drivers of the economy, with the hedging of wealth risk becoming a crucial determinant of portfolios as was the case previously in Section 3.5. In brief, once again, "capital is back" in this international context, and its allocation is of first-order importance for economic outcomes.

4.4. Counterfactuals and long-term trend

Another value of studying those questions in the framework of this paper is that it allows to perform a number of counterfactual exercises of which I say a word here.

First, the gradual decline in the average level of home bias that has been observed worldwide can be naturally captured by a decline in the degree of imperfect financial integration τ. For instance, going from $\tau = 15\%$ to $\tau = 10\%$, leads the home bias in equity holdings at $x_t = 30\%$ to decline from $HB_t = 1.5$ to $HB_t = 1.225$ on average, consistent with (or even slightly larger than) the decline documented in Coeurdacier and Rey (2013). To get a sense of magnitudes, the shares of the domestic and foreign asset in the domestic equity

[78] This also occurs for a negative shock to the foreign output, which decreases x_t for most of the state space, but increases y_t.

portfolio, $w_{h,t}/(w_{h,t}+w_{f,t}), w_{f,t}/(w_{h,t}+w_{f,t})$, go from 68% and 32%, to 55% and 45%, respectively.[79] This evolution is not only powerful in the model, but also appears realistic given that a reduction in barriers to international financial trade is likely to be one of the main determinants of the explosion in capital flows worldwide that has occurred since the 1990s.[80] Importantly, and as expected given our discussion throughout, the change in the degree of financial integration, even though it allows to broadly match the home bias in equity holdings around $x_t = 30\%$, is not without consequences for the evolution of portfolios throughout the state space. For instance, the share of the domestic portfolio invested in the domestic asset can go from large, in fact above 100% of wealth using leverage when x_t is small, to much smaller as τ decreases.[81] The change in the degree of financial integration is also reflected in the trading of the bond. Even though it remains large due to the persisting differences in tolerance for risk, bond trading becomes slightly more limited as markets become more integrated, consistent with the fact that like in Bhamra et al. (2014), an increasing share of risk sharing takes place through trading in equity assets. This prediction is consistent with the fall in bond trading that has occurred in the 1990s in G7 countries documented in Evans and Hnatkovska (2014). The share of the bond in portfolios also becomes less asymmetric as a function of the relative supply.

Better integrated markets in turn have consequences in terms of asset prices. Risk premia and Sharpe ratios decrease moderately on average, while the interest rate increases.[82]. Each becomes slightly less dependent on the wealth share, consistent with risk sharing between international investors becoming easier, even though x_t broadly remains as important a determinant as fundamentals due to the remaining asymmetries in risk tolerance. However, the consequence is most visible in terms of asset comovements: the correlation across asset returns worldwide in the model goes from 0.68 to 0.75 to 0.77 on average, for $\tau = 15, 5$ and 0%. This result, which arises naturally with the better integration of international markets and despite no correlation in fundamentals, goes some way in replicating the findings of

[79]The raw shares go from $w_{h,t} = 105\%$ and $w_{f,t} = 49\%$, to $w_{h,t} = 85\%$ and $w_{f,t} = 70\%$. Contrary to the shares in the *equity* portfolio, which sum to 100%, raw shares in the total portfolio do not because the bond is used to lever up.

[80]This is not to say that this process has been smooth: an important dimension of the Global Financial Cycle, and of the Great Recession in particular, has been a strong retrenchment in capital flows. Cf. Milesi-Ferretti and Tille (2011), and Miranda-Agrippino and Rey (2020). Those are aspects that extensions of the framework, discussed in Section 4.5, could help match as well.

[81]If τ becomes too small, or even goes towards zero, the home bias in equity holdings ultimately turns back into a foreign bias as discussed in Section 3 given that goods are good substitutes, consistent with modern estimation of θ. Even though τ is likely to have decrease significantly over time however, a world with no friction whatsoever in international financial markets remains far from realistic so that this concern remains modest in practice. Still, generating additional ways of reproducing the home bias in equity holdings that obtains in the data by introducing additional channels that could make the wealth share itself move differently and more autonomously, is an important avenue for future research. As mentioned previously, this could for instance take the form of idiosyncratic labor income or financial risk, or a time-varying labor income share against which investors would want to hedge by tilting their portfolios towards the home asset in the spirit of Coeurdacier and Gourinchas (2016).

[82]Specifically, risk premia decrease by 6.6bp (5%) on average, and Sharpe ratios by 0.0143 (4%). The interest rate increases by 9bp (3%).

Jordà et al. (2019). Indeed, the authors document a large secular increase in the synchronization of global equity markets that goes above and beyond the growing integration of real variables, and is driven in part by risk premia and changes in global risk aversion. Like for portfolios, the decrease in τ impacts not only the level but also the shape of the conditional covariance and correlation of returns. Even though $\text{corr}_t(dR_t, dR_t^*)dt^{-1}$ still tends to increase faster as the relative supply of the domestic good decreases, consistent with the remaining asymmetries in risk tolerance combined with investors having differing preferred goods, the evolution in the wealth share dimension changes significantly. While the correlation tends to increase as one of the countries become dominant in world wealth when risk sharing is imperfect, reaching a global minimum at the point of symmetry in the world economy, it tends to *in*crease when this happens under perfect risk sharing, reaching close to a global maximum at the symmetric point like in the baseline of Section 3. This underlines the interaction between asset correlation, which are related to the diversification benefits provided by the assets even though it varies a lot in this context, and the extent of risk sharing.

Even though a full discussion is omitted in the interest of space, asymmetric relaxation in the degree of international market integration is also an interesting phenomenon to look into: in practice, the financial markets of the United States have become much more accessible than that of some of its major trading patterns such as China, India, and other emerging markets. This is captured in the current framework by assuming that $\tau = 15\%$, i.e. the United States faces frictions when investing in foreign equities, while $\tau^* < \tau = 15\%$, i.e. foreign investors have a much wider access to capital markets in the United States. As expected, the amount of asymmetry in risk premia, their comovements, and portfolios, is greatly reinforced.

More generally, introducing stochastic degrees of financial integration, for instance with time-varying taxes, or micro-founding the underlying source of frictions in international markets are interesting avenues left for future research. Changes in the tax on foreign dividends, potentially asymmetric, could also be used to study the impact on global asset prices, portfolios, and risk sharing, of macroprudential policies aimed at curbing sudden international capital flows.

The model could also be used to study a number of additional counterfactual scenarios. For instance, what happens when the financial systems of other countries become more developed so that foreign investors become more able and willing to carry risk? Here, this could be captured as a decrease in the risk aversion of the foreign investor γ^* towards that of the United States $\gamma = 5$, and the economy would get closer to that studied in Section 3.5 with imperfect risk sharing but less asymmetries. In this context, introducing more than two trees and countries could prove worthwhile to study the type of phenomena that occur when the country at the center of the international financial system switches, e.g. when the world transitioned from the United Kingdom to the United States at its center in the 20th century. It could shed light on the likely impact of China or Europe becoming the new center country, or a second big player in international financial markets in a multi-polar world. The effect on asset pricing, portfolios, and risk sharing are likely to be large, given the diverging preferences of those countries, in terms of goods but also in their preference for saving *etc.*

This could also capture possible periods of instability that can occur in the transition from one hegemon to the other in the spirit of Nurkse (1944) and Farhi and Maggiori (2018). Another dimension that appears important would be to introduce several types of investors in the economy, and in particular global financial intermediaries. I discuss some such possible extensions below, and leave those promising directions for ongoing or future research.

Finally, the framework could also be used not only to study short-term high-frequency dynamics typical of crisis situations like above, but could also be reinterpreted and used at a lower frequency. In that case, it can shed light on what happens once the share of the United States in world wealth decreases as is likely to happen with the rise of emerging countries as central players of the international financial system. This can also be captured as a decrease in the wealth share of the United States in this model, and naturally leads to a decrease in world wide interest rate of the type that has been discussed in Caballero et al. (2008) or Hall (2016).

Note that, for this application to be fully analyzed, the model would have to be modified in the following sense. As is, the asymmetric position taken in the international bond, with the domestic investor borrowing for most of the state space, sometimes aggressively, leads to the expected result that in the long run, the domestic risk-tolerant investor comes to hold the majority of the wealth in the world economy. This is reminiscent of standard results in the classical literature on multi-agent asset pricing such as Dumas (1989) and comes from the fact that risky assets, to which the domestic investor allocates a larger weight in her portfolio, pay a positive premium in expectations. In this environment, this takes the form of both the drift, $\mu_{x,t}x_t$, and diffusion terms, $\sigma_{xz,t}x_t, \sigma_{xz^*,t}x_t$, of the wealth share all being positive for most of the state space, which results in an increase in the wealth share for both domestic and foreign shocks as well as in expectations (Figures F.19 and F.20). In short, the United States dominates the economy in the long run. To reverse this result, there are several possibilities.

One of them could be to introduce the fact that the output of the foreign tree, if it is meant to represent that of emerging countries, grows faster than the output in the United States. Provided that the foreign country holds a sufficiently large share of the foreign asset to benefit from it, which happens if markets are sufficiently imperfectly integrated but the asymmetry in tolerance for risk remains moderate, the foreign investor could be made to dominate the economy in the long run as the output of the foreign tree also comes to dominate. If not sufficient, this could also be combined with a reintroduction of labor income, which, by providing the foreign investor with a guaranteed share of the output of the faster-growing tree, would once again tend to push its wealth share to increase. There could also be other ways for instance using asymmetries in other dimensions of preferences such as the elasticity of intertemporal substitution, which partly governs the propensity to save of investors. Importantly, once the foreign country starts becoming larger in the long run, the global interest rate would gradually decline as has been observed empirically, with the risk-tolerant investor in the United States becoming increasingly small, in the spirit of Hall (2016) and as also discussed in Gourinchas et al. (2017). This phenomenon would happen in the long run not withstanding the fact that short-run dynamics would still be akin to those presented in Section 4.3.

Those brief illustrations show that the framework is a versatile building block to study a wealth of real-world applications in a unified context.

4.5. Extensions

Beyond those applications, the framework can be extended to capture important additional specificities of the international financial system. A couple of such extensions have been mentioned throughout the paper such as the introduction of additional countries or assets, or a more general specification of the share of labor income or taxes as being stochastic. Various ways of making the model stationary could also be interesting to explore.[83]

In addition, because I solve for the decentralized solution throughout, the framework is readily set to tackle more general market structures beyond imperfect risk sharing such as incomplete markets that would arise in the presence of idiosyncratic labor income risk as in Kaplan et al. (2018), or capital risk as in Brunnermeier and Sannikov (2014, 2015). Particularly interesting and relevant in this context will also be the addition of constraints on the portfolios of international investors, e.g. by adapting Gârleanu and Pedersen (2011), Chabakauri (2013) to my two-good international economy, which could lead to a strong reinforcement of the type of dynamics discussed in Section 4.3. Taken together, those different channels will likely lead to a strengthening of the dispersion and role of the wealth share in equilibrium.

The framework can also be extended along more ambitious dimensions. The most promising among them relate to the introduction in an international setting of the type of financial intermediaries of the type that has been discussed in the recent intermediary asset pricing literature e.g. in Daníelsson et al. (2012), He and Krishnamurthy (2013), Adrian and Shin (2014), or Adrian and Boyarchenko (2015). Those global intermediaries, which are very relevant in practice, can be involved in the dealing of foreign currencies, in the spirit of Hau and Rey (2006) and Gabaix and Maggiori (2015), or can play the role of bankers as in Maggiori (2017) and Jiang et al. (2020). As an illustration, in ongoing work (Sauzet, 2020a), I explore a third possibility: the introduction of a global asset manager. This addition is briefly described in Appendix E.1 and could help capture additional aspects of the Global Financial Cycle of Rey (2013) and Miranda-Agrippino and Rey (2020), pertaining to the leverage and role of global financial intermediaries. The combination of global financial intermediaries with time-varying demand for safe assets, which could be generated by the introduction of multiple heterogeneous investors *within* each country, could also help make way towards a resolution for the so-called "reserve currency paradox" emphasized by Maggiori (2017). I

[83] This could be done e.g. by adapting the share process of Menzly et al. (2004), Santos and Veronesi (2006) to y_t so that neither of the goods and assets dominates the economy in the long run, which could also ensure the survival of both investors. Another possibility could be to adapt the overlapping-generations structure of Gârleanu and Panageas (2015) to my multi-good international context.

briefly touch upon this question in Appendix E.2 and it is also explored in ongoing work Sauzet (2020c).

Finally, from a methodological standpoint, the number of state variables is likely to rapidly increase with those extensions. Because computationally traditional projection methods are very much subject to the curse of dimensionality, higher-dimensional methods will be required. For instance, even the addition of a third state variable, like in the global asset manager extension, renders the resolution significantly slower, and increasing the order of approximation much beyond $N = 10$ proves difficult.[84] One such method consists in naturally extending the concept of projection approaches, but to replace the Chebyshev polynomials in the approximation by neural networks, which are designed specifically for high-dimensional settings. I am developing these "projection methods via neural networks" for continuous-time models in Sauzet (2020b). I discuss them in slightly more details in Section E.3, and they should prove very useful as I pursue yet more ambitions extensions.

In summary, the framework in this paper is well-suited to handle several applications and extensions. The main application throughout Section 4 has shown that the model is able to replicate a vast number of facts about the structure and dynamics of the international financial system, and about asset returns in that context, which are strongly borne out in the data. More generally, the combination of the extensions mentioned above and higher-dimensional resolution approaches such as the "projection methods via neural networks" developed in Sauzet (2020b) provide many promising avenues for future research.

5. Conclusion

This paper has two main contributions. First, I adapt recent advances in multi-agent continuous-time asset pricing models to a two-country, two-good economy in which investors have recursive preferences and a bias in consumption towards their local good. This allows me to characterize the global solution to the international portfolio choice problem in full generality, a long-standing open issue in international finance to which the literature had only provided a piecemeal answer.

The main economic message from the first contribution is that the allocation of wealth across investors matters in a general international portfolio choice setting. This finding resonates with an emerging theme in the broader economic literature that has recently emphasized the role of the wealth distribution in determining economic outcomes in macroeconomics (e.g. Brunnermeier and Sannikov, 2014, Kaplan et al., 2018), finance (e.g. Gomez, 2017, Lettau et al., 2019, Greenwald et al., 2020), and economics more generally (e.g. Piketty and Zucman, 2014). In other words, "capital is back" in this setting too: the allocation of

[84] Finer ways to construct the Chebyshev polynomials and corresponding grids, such as complete polynomials or Smolyak's algorithm, can help. Ultimately however, they are also limited.

wealth across international investors has a prime role in driving asset prices, portfolios, and risk sharing, an aspect that had received little emphasis thus far.

The allocation of wealth matters both as a state variable that captures the average investor in the world economy and directly impacts economic outcomes, and as a pricing factor that is hedged by international investors. Its effect is relevant even in a baseline with symmetric calibration and perfect risk sharing, but grows tremendously as markets become imperfectly integrated, and as investors become more heterogeneous. The results also emphasize both (i) the state-dependence of most economic variables in this environment – e.g. portfolios vary substantially with the allocation of wealth –, and (ii) the vital impact of the calibration of preferences – e.g. the potency of imperfect financial integration is strongly reduced with a high elasticity of intertemporal substitution. This makes the novel framework presented in this paper, which is based on a global solution method and allows for general recursive preferences including asymmetries, particularly adapted to study this economy.

The framework is a well-suited building block towards several applications and extensions. My second contribution focuses on one of them and shows that the model can be used to capture a number of stylized facts about the structure and dynamics of the international financial system, and of asset returns in that context.

The introduction of asymmetries in the tolerance for risk of international investors naturally replicates the role of the United States as the world banker, documented in Gourinchas and Rey (2007b) and Gourinchas et al. (2017), and the exorbitant privilege enjoyed by the country in the form of higher excess returns. A modest degree of imperfect financial integration also generates a plausible home bias in equity holdings for both investors.

Importantly, the framework does not only replicate facts about external portfolios on average, but the asymmetry in risk tolerance also yields a number of predictions about the *dynamics* of the international financial system that are strongly borne out in the data. As a crisis hits, the center country is impacted particularly severely due to its high allocation to risky assets, so that it transfers a large amount of wealth to the rest of the world. This exorbitant duty is the flip side of its exorbitant privilege in normal times: the United States must become the world insurer in times of trouble. In addition, by worsening the wealth position of the risk-tolerant world banker, the shock leads to a sharp increase in global risk aversion, which in turn pushes up all risk premia and Sharpe ratios worldwide. These two markers are reminiscent of some aspects of the Global Financial Cycle of Rey (2013), and Miranda-Agrippino and Rey (2020), for which a general equilibrium exploration had remained elusive. Those patterns are representative of the type of global risk-off scenarios that typically occur in times of global crisis such as most recently in the Great Recession of 2008 or the Global Pandemic of 2020.

The model can also shed light on the reaction of global portfolios to shocks, and the process of external adjustment of the center country, emphasizing the primordial role played by valuation effects in this context. It also allows to run a number of counterfactual exercises.

From an asset pricing perspective, the specialization of the model also speaks to a number of facts about asset returns dynamics in this international environment. Namely, risk premia, Sharpe ratios – and to some extent volatilities and correlations in a relevant region of the state space – are all countercyclical in the sense that they increase following the shock, consistent with a wide range of evidence notably for the United States. Importantly, those patterns are driven for a large part not by changes in the quantity of risk but by the evolution of the compensation for risk, captured here by the time-varying global risk aversion. This is in line with a large literature that has seen changes in the price of risk emerge as a crucial explanation behind asset return predictability more generally.

In summary of the second contribution, a seemingly small change in the specification of the model – the introduction of asymmetries in risk tolerance – generates a vast number of facts about the structure and dynamics of the international financial system, and about asset returns, which are strongly borne out in the data.

The model is also a well-suited building block for many potential extensions. The most promising among them are related to the introduction in an international setting of financial intermediaries of the type that has been discussed in the recent intermediary asset pricing literature, and illustrations were briefly discussed in Section 4.5 e.g. with the inclusion of a global asset manager (Sauzet, 2020a). The implementation of those extensions will likely require higher-dimensional methods such as the "projection methods via neural networks" being developed in Sauzet (2020b). I leave all these promising avenues for future research.

References

Adrian, Tobias, and Nina Boyarchenko, 2015, Intermediary Balance Sheets, 2015 Meeting Papers 239, Society for Economic Dynamics.

Adrian, Tobias, and Hyun Song Shin, 2014, Procyclical leverage and value-at-risk, *Review of Financial Studies* 27, 373–403.

Anderson, Evan W., 2005, The dynamics of risk-sensitive allocations, *Journal of Economic Theory* 125, 93 – 150.

Anderson, Robert M., and Roberto C. Raimondo, 2008, Equilibrium in continuous-time financial markets: Endogenously dynamically complete markets, *Econometrica* 76, 841–907.

Backus, David K., and Gregor W. Smith, 1993, Consumption and real exchange rates in dynamic economies with non-traded goods, *Journal of International Economics* 35, 297 – 316.

Bansal, Ravi, and Amir Yaron, 2004, Risks for the long run: A potential resolution of asset pricing puzzles, *The Journal of Finance* 59, 1481–1509.

Basak, Suleyman, and Domenico Cuoco, 1998, An Equilibrium Model with Restricted Stock Market Participation, *The Review of Financial Studies* 11, 309–341.

Basak, Suleyman, and Michael Gallmeyer, 2003, Capital Market Equilibrium with Differential Taxation, *Review of Finance* 7, 121–159.

Baxter, Marianne, and Urban Jermann, 1997, The international diversification puzzle is worse than you think, *American Economic Review* 87, 170–80.

Baxter, Marianne, Urban J Jermann, and Robert G King, 1998, Nontraded goods, nontraded factors, and international non-diversification, *Journal of International Economics* 44, 211 – 229.

Bhamra, Harjoat S., Nicolas Coeurdacier, and Stéphane Guibaud, 2014, A dynamic equilibrium model of imperfectly integrated financial markets, *Journal of Economic Theory* 154, 490 – 542.

Brandt, Michael W., John H. Cochrane, and Pedro Santa-Clara, 2006, International risk sharing is better than you think, or exchange rates are too smooth, *Journal of Monetary Economics* 53, 671 – 698.

Brandt, Michael W., and Qiang Kang, 2004, On the relationship between the conditional mean and volatility of stock returns: A latent var approach, *Journal of Financial Economics* 72, 217–257.

Brunnermeier, Markus, and Yuliy Sannikov, 2014, A macroeconomic model with a financial sector, *American Economic Review* 104, 379–421.

Brunnermeier, Markus K., and Lasse Heje Pedersen, 2009, Market liquidity and funding liquidity, *The Review of Financial Studies* 22, 2201–2238.

Brunnermeier, Markus K., and Yuliy Sannikov, 2015, International credit flows and pecuniary externalities, *American Economic Journal: Macroeconomics* 7, 297–338.

Brunnermeier, Markus K., and Yuliy Sannikov, 2019, International monetary theory: A risk perspective, *Working Paper* .

Caballero, Ricardo J., Emmanuel Farhi, and Pierre-Olivier Gourinchas, 2008, An equilibrium model of "global imbalances" and low interest rates, *American Economic Review* 98, 358–93.

Camanho, Nelson, Harald Hau, and Hélène Rey, 2018, Global portfolio rebalancing and exchange rates, Working paper.

Campbell, John Y., 1999, Chapter 19 asset prices, consumption, and the business cycle, volume 1 of *Handbook of Macroeconomics*, 1231 – 1303 (Elsevier).

Campbell, Sean D., and Francis X. Diebold, 2009, Stock returns and expected business conditions: Half a century of direct evidence, *Journal of Business & Economic Statistics* 27, 266–278.

Cass, David, and Anna Pavlova, 2004, On trees and logs, *Journal of Economic Theory* 116, 41 – 83.

Chabakauri, Georgy, 2013, Dynamic Equilibrium with Two Stocks, Heterogeneous Investors, and Portfolio Constraints, *The Review of Financial Studies* 26, 3104–3141.

Chan, Yeung Lewis, and Leonid Kogan, 2002, Catching up with the joneses: Heterogeneous preferences and the dynamics of asset prices, *Journal of Political Economy* 110, 1255–1285.

Cochrane, John H., Francis A. Longstaff, and Pedro Santa-Clara, 2008, Two Trees, *The Review of Financial Studies* 21, 347–385.

Coeurdacier, Nicolas, 2009, Do trade costs in goods market lead to home bias in equities?, *Journal of International Economics* 77, 86 – 100.

Coeurdacier, Nicolas, and Pierre-Olivier Gourinchas, 2016, When bonds matter: Home bias in goods and assets, *Journal of Monetary Economics* 82, 119 – 137.

Coeurdacier, Nicolas, and Hélène Rey, 2013, Home bias in open economy financial macroeconomics, *Journal of Economic Literature* 51, 63–115.

Coeurdacier, Nicolas, Hélène Rey, and Pablo Winant, 2020, Financial integration and growth in a risky world, *Journal of Monetary Economics* 112, 1 – 21.

Colacito, Ric, Max Croce, Steven Ho, and Philip Howard, 2018, Bkk the ez way: International long-run growth news and capital flows, *American Economic Review* 108, 3416–49.

Colacito, Riccardo, and Mariano M. Croce, 2011, Risks for the long run and the real exchange rate, *Journal of Political Economy* 119, 153–181.

Colacito, Riccardo, and Mariano M. Croce, 2013, International asset pricing with recursive preferences, *Journal of Finance* 68, 2651–2686.

Cole, Harold L., and Maurice Obstfeld, 1991, Commodity trade and international risk sharing: How much do financial markets matter?, *Journal of Monetary Economics* 28, 3–24.

Coppola, Antonio, Matteo Maggiori, Brent Neiman, and Jesse Schreger, 2020, Redrawing the map of global capital flows: The role of cross-border financing and tax havens, Working Paper 26855, National Bureau of Economic Research.

Corsetti, Giancarlo, Luca Dedola, and Sylvain Leduc, 2008, International Risk Sharing and the Transmission of Productivity Shocks, *The Review of Economic Studies* 75, 443–473.

Cuoco, Domenico, and Hua He, 1994, Dynamic equilibrium in infinite-dimensional economies with incomplete financial markets, Working paper, UPenn.

Daníelsson, Jón, Hyun Song Shin, and Jean-Pierre Zigrand, 2012, Procyclical leverage and endogenous risk, Mimeo, LSE, http://www.riskresearch.org.

Devereux, Michael B., and Alan Sutherland, 2011, Country portfolios in open economy macro-models, *Journal of the European Economic Association* 9, 337–369.

Drechsler, Itamar, Alexi Savov, and Philipp Schnabl, 2018, A model of monetary policy and risk premia, *The Journal of Finance* 73, 317–373.

Duarte, Victor, 2019, Machine learning for continuous-time economics, *Available at SSRN: https://ssrn.com/abstract=3012602* .

Duffie, D., J. Geanakoplos, A. Mas-Colell, and A. McLennan, 1994, Stationary markov equilibria, *Econometrica* 62, 745–781.

Duffie, Darrell, and Larry Epstein, 1992, Stochastic differential utility, *Econometrica* 60, 353–94.

Dumas, Bernard, 1989, Two-Person Dynamic Equilibrium in the Capital Market, *The Review of Financial Studies* 2, 157–188.

Dumas, Bernard, 1992, Dynamic Equilibrium and the Real Exchange Rate in a Spatially Separated World, *The Review of Financial Studies* 5, 153–180.

Dumas, Bernard, Raman Uppal, and Tan Wang, 2000, Efficient intertemporal allocations with recursive utility, *Journal of Economic Theory* 93, 240–259.

Ehling, Paul, and Christian Heyerdahl-Larsen, 2015, Complete and incomplete financial markets in multi-good economies, *Journal of Economic Theory* 160, 438 – 462.

Evans, Martin D.D., and Viktoria Hnatkovska, 2012, A method for solving general equilibrium models with incomplete markets and many financial assets, *Journal of Economic Dynamics and Control* 36, 1909 – 1930.

Evans, Martin D.D., and Viktoria V. Hnatkovska, 2014, International capital flows, returns and world financial integration, *Journal of International Economics* 92, 14–33.

Fama, Eugene F., and Kenneth R. French, 1989, Business conditions and expected returns on stocks and bonds, *Journal of Financial Economics* 25, 23 – 49.

Fang, Xiang, 2019, Intermediary leverage and currency risk premium, Working paper.

Farhi, Emmanuel, and Matteo Maggiori, 2018, A Model of the International Monetary System*, *The Quarterly Journal of Economics* 133, 295–355.

Fernández-Villaverde, Jesús, and Lawrence Christiano, 2011, Nber summer institute, computational tools & macroeconomic applications, Methods lectures.

Ferson, Wayne E., and Campbell R. Harvey, 1991, The variation of economic risk premiums, *Journal of Political Economy* 99, 385–415.

French, Kenneth, and James Poterba, 1991, Investor diversification and international equity markets, *American Economic Review* 81, 222–26.

Gabaix, Xavier, and Ralph S.J. Koijen, 2020, Granular instrumental variables, *Available at SSRN: https://ssrn.com/abstract=3368612.* .

Gabaix, Xavier, and Matteo Maggiori, 2015, International liquidity and exchange rate dynamics*, *The Quarterly Journal of Economics* 130, 1369–1420.

Gârleanu, Nicolae, and Stavros Panageas, 2015, Young, old, conservative, and bold: The implications of heterogeneity and finite lives for asset pricing, *Journal of Political Economy* 123, 670 – 685.

Gârleanu, Nicolae, Stavros Panageas, and Jianfeng Yu, 2020, Impediments to Financial Trade: Theory and Applications, *The Review of Financial Studies* 33, 2697–2727.

Gârleanu, Nicolae, and Lasse Heje Pedersen, 2011, Margin-based asset pricing and deviations from the law of one price, *The Review of Financial Studies* 24, 1980–2022.

Geanakoplos, John, 1990, An introduction to general equilibrium with incomplete asset markets, *Journal of Mathematical Economics* 19, 1 – 38.

Geanakoplos, John, and Andreu Mas-Colell, 1989, Real indeterminacy with financial assets, *Journal of Economic Theory* 47, 22 – 38.

Geanakoplos, John D, and Heraklis M Polemarchakis, 1986, Existence, regularity, and constrained suboptimality of competitive allocations, *Essays in Honor of Kenneth J. Arrow: Volume 3, Uncertainty, Information, and Communication* 3, 65.

Gehrig, Thomas, 1993, An information based explanation of the domestic bias in international equity investment, *The Scandinavian Journal of Economics* 95, 97–109.

Gomez, Matthieu, 2017, Asset prices and wealth inequality, 2017 Meeting Papers 1155, Society for Economic Dynamics.

Gourinchas, Pierre-Olivier, and Hélène Rey, 2007a, International financial adjustment, *Journal of Political Economy* 115, 665–703.

Gourinchas, Pierre-Olivier, and Hélène Rey, 2007b, From World Banker to World Venture Capitalist: U.S. External Adjustment and the Exorbitant Privilege, in *G7 Current Account Imbalances: Sustainability and Adjustment*, NBER Chapters, 11–66 (National Bureau of Economic Research, Inc).

Gourinchas, Pierre-Olivier, Hélène Rey, and Nicolas Govillot, 2017, Exorbitant Privilege and Exorbitant Duty, Working paper, University of California, Berkeley, LBS.

Gourinchas, Pierre-Olivier, Hélène Rey, and Maxime Sauzet, 2019, The international monetary and financial system, *Annual Review of Economics* 11, 859–893.

Greenwald, Daniel L, Martin Lettau, and Sydney C Ludvigson, 2020, How the wealth was won: Factors shares as market fundamentals, Working Paper 25769, NBER.

Hall, Robert E., 1988, Intertemporal substitution in consumption, *Journal of Political Economy* 96, 339–357.

Hall, Robert E, 2016, The role of the growth of risk-averse wealth in the decline of the safe real interest rate, Working Paper 22196, NBER.

Hansen, Lars Peter, Paymon Khorrami, and Fabrice Tourre, 2018, Comparative valuation dynamics in models with financing restrictions .

Harrison, Paul, and Harold H. Zhang, 1999, An investigation of the risk and return relation at long horizons, *The Review of Economics and Statistics* 81, 399–408.

Hart, Oliver D, 1975, On the optimality of equilibrium when the market structure is incomplete, *Journal of Economic Theory* 11, 418 – 443.

Harvey, Campbell R., 2001, The specification of conditional expectations, *Journal of Empirical Finance* 8, 573 – 637, On the Predictability of Asset Returns.

Hassan, Tarek, 2013, Country size, currency unions, and international asset returns, *The Journal of Finance* 68, 2269–2308.

Hau, Harald, and Hélène Rey, 2006, Exchange Rates, Equity Prices, and Capital Flows, *The Review of Financial Studies* 19, 273–317.

He, Zhiguo, and Arvind Krishnamurthy, 2013, Intermediary Asset Pricing, *American Economic Review* 103, 732–770.

Heathcote, Jonathan, and Fabrizio Perri, 2002, Financial autarky and international business cycles, *Journal of Monetary Economics* 49, 601–627.

Hugonnier, J., S. Malamud, and E. Trubowitz, 2012, Endogenous completeness of diffusion driven equilibrium markets, *Econometrica* 80, 1249–1270.

Imbs, Jean, and Isabelle Méjean, 2015, Elasticity optimism, *American Economic Journal: Macroeconomics* 7, 43–83.

Jeanne, Olivier, and Andrew K. Rose, 2002, Noise Trading and Exchange Rate Regimes*, *The Quarterly Journal of Economics* 117, 537–569.

Jiang, Zhengyang, Arvind Krishnamurthy, and Hanno Lustig, 2020, Dollar safety and the global financial cycle, Working Paper 27682, NBER.

Jordà, Òscar, Moritz Schularick, Alan Taylor, and Felix Ward, 2019, Global financial cycles and risk premiums, *IMF Economic Review* 67, 109–150.

Jordà, Òscar, Moritz Schularick, and Alan M. Taylor, 2016, *Macrofinancial History and the New Business Cycle Facts*, 213–263 (University of Chicago Press).

Judd, Kenneth, 1992, Projection methods for solving aggregate growth models, *Journal of Economic Theory* 58, 410–452.

Judd, Kenneth, 1998, *Numerical Methods in Economics*, volume 1, first edition (The MIT Press).

Kaplan, Greg, Benjamin Moll, and Giovanni L. Violante, 2018, Monetary policy according to hank, *American Economic Review* 108, 697–743.

Kargar, Mahyar, 2019, Heterogeneous intermediary asset pricing, *Working Paper* .

Kekre, Rohan, and Moritz Lenel, 2020, Dollar demand and international risk sharing, Working paper.

Koijen, Ralph S. J, and Motohiro Yogo, 2020, Exchange rates and asset prices in a global demand system, Working Paper 27342, National Bureau of Economic Research.

Kollmann, Robert, 1995, Consumption, real exchange rates and the structure of international asset markets, *Journal of International Money and Finance* 14, 191–211.

Kollmann, Robert, 2006, A dynamic equilibrium model of international portfolio holdings: Comment, *Econometrica* 74, 269–273.

Kouri, Pentti, 1982, Balance of Payment and the Foreign Exchange Market: A Dynamic Partial Equilibrium Model, *Economic interdependence and Flexible Exchange Rates ed. Jagdeep Bhandari and Bluford Putnam* 116–56.

Kubler, Felix, and Karl Schmedders, 2003, Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral, *Econometrica* 71, 1767–1793.

Lettau, Martin, and Sydney C. Ludvigson, 2010, Chapter 11 - measuring and modeling variation in the risk-return trade-off, in Yacine Aït-Sahalia, and Lars Peter Hansen, eds., *Handbook of Financial Econometrics: Tools and Techniques*, volume 1 of *Handbooks in Finance*, 617 – 690 (North-Holland, San Diego).

Lettau, Martin, Sydney C. Ludvigson, and Sai Ma, 2019, Capital share risk in u.s. asset pricing, *The Journal of Finance* 74, 1753–1792.

Lucas, Robert, 1978, Asset prices in an exchange economy, *Econometrica* 46, 1429–45.

Maggiori, Matteo, 2017, Financial intermediation, international risk sharing, and reserve currencies, *American Economic Review* 107, 3038–71.

Maggiori, Matteo, Brent Neiman, and Jesse Schreger, 2020, International currencies and capital allocation, *Journal of Political Economy* 128, 2019–2066.

Martin, Ian, 2011, The forward premium puzzle in a two-country world, Working Paper 17564, National Bureau of Economic Research.

Martin, Ian, 2013, The lucas orchard, *Econometrica* 81, 55–111.

Mehra, Rajnish, and Edward C. Prescott, 1985, The equity premium: A puzzle, *Journal of Monetary Economics* 15, 145 – 161.

Menzly, Lior, Tano Santos, and Pietro Veronesi, 2004, Understanding predictability, *Journal of Political Economy* 112, 1–47.

Mertens, Thomas M., and Kenneth L. Judd, 2018, Solving an incomplete markets model with a large cross-section of agents, *Journal of Economic Dynamics and Control* 91, 349 – 368, Special Issue in Honour of Prof. Carl Chiarella.

Merton, Robert, 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867–87.

Milesi-Ferretti, Gian-Maria, and Cédric Tille, 2011, The great retrenchment: international capital flows during the global financial crisis, *Economic Policy* 26, 285–342.

Miranda, Mario J, and Paul L Fackler, 2004, *Applied computational economics and finance*.

Miranda-Agrippino, Silvia, and Hélène Rey, 2020, U.S. Monetary Policy and the Global Financial Cycle, *The Review of Economic Studies* rdaa019.

Nurkse, Ragnar, 1944, *International Currency Experience: Lessons of the Inter-war Period*, Series of League of Nations Publications. 2. Economic and financial. 1944. 2 (UN).

Obstfeld, Maurice, 2004, External adjustment, *Review of World Economics (Weltwirtschaftliches Archiv)* 140, 541–568.

Obstfeld, Maurice, 2007, International risk sharing and the costs of trade, *The Ohlin Lectures* Stockholm School of Economics.

Obstfeld, Maurice, and Kenneth Rogoff, 2001, The Six Major Puzzles in International Macroeconomics: Is There a Common Cause?, in *NBER Macroeconomics Annual 2000, Volume 15*, NBER Chapters, 339–412 (NBER, Inc).

Parra-Alvarez, Juan, 2018, A comparison of numerical methods for the solution of continuous-time dsge models, *Macroeconomic Dynamics* 22, 1555–1583.

Pavlova, Anna, and Roberto Rigobon, 2007, Asset Prices and Exchange Rates, *The Review of Financial Studies* 20, 1139–1180.

Pavlova, Anna, and Roberto Rigobon, 2008, The Role of Portfolio Constraints in the International Propagation of Shocks, *The Review of Economic Studies* 75, 1215–1256.

Pavlova, Anna, and Roberto Rigobon, 2010, An asset-pricing view of external adjustment, *Journal of International Economics* 80, 144 – 156, Special Issue on International Macro-Finance.

Piketty, Thomas, and Gabriel Zucman, 2014, Capital is Back: Wealth-Income Ratios in Rich Countries 1700–2010*, *The Quarterly Journal of Economics* 129, 1255–1310.

Polemarchakis, Heraklis M, 1988, Portfolio choice, exchange rates, and indeterminacy, *Journal of Economic Theory* 46, 414–421.

Rabitsch, Katrin, Serhiy Stepanchuk, and Viktor Tsyrennikov, 2015, International portfolios: A comparison of solution methods, *Journal of International Economics* 97, 404 – 422.

Rey, Hélène, 2013, Dilemma not trilemma: the global cycle and monetary policy independence, *Proceedings - Economic Policy Symposium - Jackson Hole* 1–2.

Santos, Tano, and Pietro Veronesi, 2006, Labor Income and Predictable Stock Returns, *The Review of Financial Studies* 19, 1–44.

Sauzet, Maxime, 2020a, Global asset manager and the global financial cycle, Work in progress.

Sauzet, Maxime, 2020b, Projection methods via neural networks for continuous-time models, Working paper.

Sauzet, Maxime, 2020c, Demand for bonds, global currency dealer, and the reserve currency paradox, Work in progress.

Schorfheide, Frank, Dongho Song, and Amir Yaron, 2018, Identifying long-run risks: A bayesian mixed-frequency approach, *Econometrica* 86, 617–654.

Schroder, Mark, and Costis Skiadas, 1999, Optimal consumption and portfolio selection with stochastic differential utility, *Journal of Economic Theory* 89, 68 – 126.

Schwert, G. William, 1989, Why does stock market volatility change over time?, *The Journal of Finance* 44, 1115–1153.

Stathopoulos, Andreas, 2017, Asset Prices and Risk Sharing in Open Economies, *The Review of Financial Studies* 30, 363–415.

Stepanchuk, Serhiy, and Viktor Tsyrennikov, 2015, Portfolio and welfare consequences of debt market dominance, *Journal of Monetary Economics* 74, 89 – 101.

Stulz, René M., 1983, The demand for foreign bonds, *Journal of International Economics* 15, 225 – 238.

Tille, Cédric, 2001, The role of consumption substitutability in the international transmission of monetary shocks, *Journal of International Economics* 53, 421 – 444, Symposium on the Japanese economic slump of the 1990's.

Tille, Cédric, and Eric van Wincoop, 2010, International capital flows, *Journal of International Economics* 80, 157–175.

Uribe, Martin, and Stephanie Schmitt-Grohé, 2017, *Open economy macroeconomics* (Princeton University Press).

Wang, Jiang, 1996, The term structure of interest rates in a pure exchange economy with heterogeneous investors, *Journal of Financial Economics* 41, 75 – 110.

Weinbaum, David, 2009, Investor heterogeneity, asset pricing and volatility dynamics, *Journal of Economic Dynamics and Control* 33, 1379 – 1397.

Appendix

A. Additional equations and results

A.1. Drift and diffusion terms for any variable

Remark A.1. *By Itô's Lemma, the geometric drift and diffusion term for any function $g_t = g(X_t)$ are given by:*

$$\frac{dg_t}{g_t} = \frac{dg(X_t)}{g(X_t)} \equiv \mu_{g,t}dt + \sigma_{g,t}^T d\vec{Z}_t \tag{A.1}$$

where:

$$\mu_{g,t} = \frac{g_{x,t}}{g_t} x_t \mu_{x,t} + \frac{g_{y,t}}{g_t} y_t \mu_{y,t} + \frac{1}{2}\frac{g_{xx,t}}{g_t} x_t^2 \sigma_{x,t}^T \sigma_{x,t} + \frac{1}{2}\frac{g_{yy,t}}{g_t} y_t^2 \sigma_{y,t}^T \sigma_{y,t} + \frac{g_{xy,t}}{g_t} x_t y_t \sigma_{x,t}^T \sigma_{y,t} \tag{A.2}$$

$$\sigma_{g,t} = \frac{g_{x,t}}{g_t} x_t \sigma_{x,t} + \frac{g_{y,t}}{g_t} y_t \sigma_{y,t} \tag{A.3}$$

This result is used repeatedly throughout the paper.

As a point of notation, recall that for any function g, g_t simply denotes $g(X_t)$, not the time-derivative of g (which is zero because the model is stationary due to infinite horizon). $g_{x,t}, g_{y,t}, g_{xx,t}, g_{yy,t}, g_{xy,t}$ denote the partial derivatives of $g(X_t)$.

A.2. Returns, and risk premia

The (geometric) drifts and diffusion terms for asset returns are obtained from Itô's Lemma and are as follows

$$\begin{aligned} dR_t &= \mu_{R,t}dt + \sigma_{R,t}^T d\vec{Z}_t \\ &\equiv \Big(F_t + \mu_{p,t} + \mu_Y + \sigma_{p,t}^T \sigma_Y - \mu_{F,t} + \sigma_{F,t}^T \sigma_{F,t} - (\sigma_{p,t} + \sigma_Y)^T \sigma_{F,t} \Big) dt \\ &\quad + (\sigma_{p,t} + \sigma_Y - \sigma_{F,t})^T d\vec{Z}_t \end{aligned} \tag{A.4}$$

$$\begin{aligned} dR_t^* &= \mu_{R*,t}dt + \sigma_{R*,t}^T d\vec{Z}_t \\ &\equiv \Big(F_t^* + \mu_{p*,t} + \mu_Y^* + \sigma_{p*,t}^T \sigma_Y^* - \mu_{F*,t} + \sigma_{F*,t}^T \sigma_{F*,t} - (\sigma_{p*,t} + \sigma_Y^*)^T \sigma_{F*,t} \Big) dt \\ &\quad + (\sigma_{p*,t} + \sigma_Y^* - \sigma_{F*,t})^T d\vec{Z}_t \end{aligned} \tag{A.5}$$

where $\mu_{p,t}$, $\mu_{p*,t}$, $\mu_{F,t}$, $\mu_{F*,t}$, $\sigma_{p,t}$, $\sigma_{p*,t}$, $\sigma_{F,t}$, $\sigma_{F*,t}$ are obtained using Remark A.1 above.

Proposition A.1. *The expected risk premia on the equity assets are given by*

$$\begin{aligned}
\mu_{R,t} - r_t = & \gamma_t \sigma_{R,t}^T \left\{ z_t \sigma_{R,t} + (1 - z_t) \sigma_{R^*,t} \right\} \\
& - \gamma_t \sigma_{R,t}^T \sigma_{x,t} x_t \left\{ x_t \left(\frac{1}{\gamma} \right) \left(\frac{1-\gamma}{1-\psi} \right) \frac{J_{x,t}}{J_t} + (1 - x_t) \left(\frac{1}{\gamma^*} \right) \left(\frac{1-\gamma^*}{1-\psi^*} \right) \frac{J_{x,t}^*}{J_t^*} \right\} \\
& - \gamma_t \sigma_{R,t}^T \sigma_{y,t} y_t \left\{ x_t \left(\frac{1}{\gamma} \right) \left(\frac{1-\gamma}{1-\psi} \right) \frac{J_{y,t}}{J_t} + (1 - x_t) \left(\frac{1}{\gamma^*} \right) \left(\frac{1-\gamma^*}{1-\psi^*} \right) \frac{J_{y,t}^*}{J_t^*} \right\} \\
& + \gamma_t \left(\frac{1 - x_t}{\gamma^*} \right) \tau^* F_t
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
\mu_{R^*,t} - r_t = & \gamma_t \sigma_{R^*,t}^T \left\{ z_t \sigma_{R,t} + (1 - z_t) \sigma_{R^*,t} \right\} \\
& - \gamma_t \sigma_{R^*,t}^T \sigma_{x,t} x_t \left\{ x_t \left(\frac{1}{\gamma} \right) \left(\frac{1-\gamma}{1-\psi} \right) \frac{J_{x,t}}{J_t} + (1 - x_t) \left(\frac{1}{\gamma^*} \right) \left(\frac{1-\gamma^*}{1-\psi^*} \right) \frac{J_{x,t}^*}{J_t^*} \right\} \\
& - \gamma_t \sigma_{R^*,t}^T \sigma_{y,t} y_t \left\{ x_t \left(\frac{1}{\gamma} \right) \left(\frac{1-\gamma}{1-\psi} \right) \frac{J_{y,t}}{J_t} + (1 - x_t) \left(\frac{1}{\gamma^*} \right) \left(\frac{1-\gamma^*}{1-\psi^*} \right) \frac{J_{y,t}^*}{J_t^*} \right\} \\
& + \gamma_t \left(\frac{x_t}{\gamma} \right) \tau F_t^*
\end{aligned} \tag{A.7}$$

where $\gamma_t \equiv \left(\frac{x_t}{\gamma} + \frac{1-x_t}{\gamma^*} \right)^{-1}$ *is the wealth-weighted global risk aversion.*

A.3. Foreign investor problem

The representative consumer of the foreign country solves:

$$V_t^* = \max_{\{C_{h,u}^*, C_{f,u}^*, w_{h,u}^*, w_{f,u}^*\}_{u=t}^{\infty}} \mathbb{E}_t \left[\int_t^{\infty} f\left(C_u^*, V_u^*\right) du \right] \tag{A.8}$$

$$f(C^*, V^*) \equiv \left(\frac{1-\gamma^*}{1-1/\psi^*} \right) V^* \left[\left(\frac{C^*}{\left[(1-\gamma^*)V^*\right]^{1/(1-\gamma^*)}} \right)^{1-1/\psi^*} - \rho^* \right] \tag{A.9}$$

subject to:

$$\begin{aligned}
\frac{dW_t^*}{W_t^*} = & \left(r_t + w_{h,t}^* \left(\mu_{R,t} - r_t \right) + w_{f,t}^* \left(\mu_{R^*,t} - r_t \right) - P_t^* c_t^* \right) dt \\
& + \left(w_{h,t}^* \sigma_{R,t} + w_{f,t}^* \sigma_{R^*,t} \right)^T d\vec{z}_t
\end{aligned} \tag{A.10}$$

$$C_t^* = \left[(1-\alpha)^{\frac{1}{\theta}} C_{h,t}^{*\frac{\theta-1}{\theta}} + \alpha^{\frac{1}{\theta}} C_{f,t}^{*\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \tag{A.11}$$

All parameters can differ from those of the domestic investor. Cf. the main text for a discussion. To complete the definition of the optimization problem, the investor is subject to a standard transversality condition, and W_0^* is given. Note also that $W_t^* \geqslant 0$.

A.4. Equilibrium

The definition of the equilibrium is standard.

Definition 1. *A competitive equilibrium is a set of aggregate stochastic processes adapted to the filtration generated by $\vec{Z}$: the price of the equity asset (Q_t, Q_t^*), and the interest rate (r_t), together with a set of individual stochastic processes for each investor: consumption of each good $(C_{h,t}, C_{f,t}, C^*_{h,t}, C^*_{f,t})$, wealth (W_t, W_t^*), and portfolio shares $(w_{h,t}, w_{f,t}, w^*_{h,t}, w^*_{f,t})$, such that, given the output of the two endowment trees (Y_t, Y_t^*):*

1. *Given the aggregate stochastic processes, individual choices solve the investor optimization problem given above.*
2. *Markets clear.*
   a) *Good markets:*
$$\begin{aligned} C_{h,t} + C_{h,t} &= Y_t \\ C_{f,t} + C^*_{f,t} &= Y_t^* \end{aligned} \tag{A.12}$$
   b) *Equity markets:*
$$\begin{aligned} w_{h,t}W_t + w^*_{h,t}W_t^* &= Q_t \\ w_{f,t}W_t + w^*_{f,t}W_t^* &= Q_t^* \end{aligned} \tag{A.13}$$

Most importantly, as shown in Section 2.3 of the main text, the equilibrium can be recast as a stationary recursive Markovian equilibrium in which all variables of interest are expressed as a function of a pair of state variables $X_t \equiv (x_t, y_t)'$, whose dynamics are also solely a function of X_t. x_t is the wealth share of the domestic investor, and y_t is the relative supply of the domestic good.

A.5. Hamilton-Jacobi-Bellman equations

Proposition A.2. *J_t satisfies the Hamilton-Jacobi-Bellman equation:*
$$\begin{aligned} 0 = &\left(\frac{1}{\psi - 1}\right) P_t^{1-\psi} J_t - \left(\frac{1}{1-1/\psi}\right)\rho + r_t + \frac{\gamma}{2}\left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R^*,t}\right) \\ &+ \left(\frac{1}{1-\psi}\right)\mu_{J,t} + \frac{1}{2}\left(\frac{1}{1-\psi}\right)\left(\frac{\psi-\gamma}{1-\psi}\right)\sigma_{J,t}^T\sigma_{J,t} \end{aligned} \tag{A.14}$$

where $\mu_{J,t}, \sigma_{J,t}$ are the geometric drift and diffusion terms of J_t obtained as in Remark A.1:
$$\frac{dJ_t}{J_t} \equiv \mu_{J,t}dt + \sigma_{J,t}^T d\vec{Z}_t \tag{A.15}$$

J_t^* satisfies a similar Hamilton-Jacobi-Bellman equation.

A.6. Consumptions, goods prices

Proposition A.3. *The consumption of the home investor is given by:*

$$c_t \equiv \frac{C_t}{W_t} = P_t^{-\psi} J_t \tag{A.16}$$
$$c_{h,t} = \alpha \left(\frac{p_t}{P_t}\right)^{-\theta} c_t$$
$$c_{f,t} = (1-\alpha) \left(\frac{p_t^*}{P_t}\right)^{-\theta} c_t$$
$$P_t = \left[\alpha p_t^{1-\theta} + (1-\alpha) p_t^{*1-\theta}\right]^{1/(1-\theta)}$$

The consumption of the foreign investor is given by:

$$c_t^* \equiv \frac{C_t^*}{W_t^*} = P_t^{*-\psi^*} J_t^* \tag{A.17}$$
$$c_{h,t}^* = (1-\alpha) \left(\frac{p_t}{P_t^*}\right)^{-\theta} c_t^* \tag{A.18}$$
$$c_{f,t}^* = \alpha \left(\frac{p_t^*}{P_t^*}\right)^{-\theta} c_t^* \tag{A.19}$$
$$P_t^* = \left[(1-\alpha) p_t^{1-\theta} + \alpha p_t^{*1-\theta}\right]^{1/(1-\theta)} \tag{A.20}$$

Proposition A.4. *The terms of trade, $q_t = q(X_t)$, solves the following non-linear equation:*

$$q_t = S_t^{1/\theta} \left(\frac{y_t}{1-y_t}\right)^{1/\theta} \tag{A.21}$$

where:

$$S_t = \frac{(1-\alpha) P_t^{\theta-\psi} J_t x_t + \alpha P_t^{*\theta-\psi^*} J_t^* (1-x_t)}{\alpha J_t x_t P_t^{\theta-\psi} + (1-\alpha) P_t^{*\theta-\psi^*} J_t^* (1-x_t)} \tag{A.22}$$

Using the defintion of the numéraire, prices follow:

$$p_t = \left(a + (1-a) q_t^{1-\theta}\right)^{1/(\theta-1)} \tag{A.23}$$
$$p_t^* = p_t q_t = \left(a q_t^{\theta-1} + (1-a)\right)^{1/(\theta-1)} \tag{A.24}$$
$$P_t = \left[\alpha p_t^{1-\theta} + (1-\alpha) p_t^{*1-\theta}\right]^{1/(1-\theta)} \tag{A.25}$$
$$P_t^* = \left[(1-\alpha) p_t^{1-\theta} + \alpha p_t^{*1-\theta}\right]^{1/(1-\theta)} \tag{A.26}$$
$$\mathcal{E}_t = P_t^* / P_t \tag{A.27}$$

A.7. Calibration

This section provides details on the baseline symmetric calibration of Assumption 1.

At $\gamma = \gamma^* = 15$, risk aversion is a bit on the high side, although within the range of values that are common in asset pricing. This allows to generate slightly more realistic risk premia, given that the model only features mild frictions in the form of imperfect financial integration. (This is nothing but the equity premium puzzle of Mehra and Prescott (1985).) The risk aversion could be increased much further for the purpose of matching risk premia more closely to the data, given that recursive preferences decouple it from the inverse of the elasticity of intertemporal substitution. However, the focus in this paper is on the mechanisms rather than on an exact quantitative match. Moving forward, extensions of the model, some of which discussed in Section 4.5, will be the prime way to generate higher risk premia. Prominent examples include the introduction of portfolio constraints, and of non-diversifiable idiosyncratic risk.

Although the elasticity of intertemporal substitution is set at $\psi = \psi^* = 2$ in the baseline, consistent with recent estimates e.g. in Schorfheide et al. (2018) and with values around $\psi = 1.5$ that have been used in the asset pricing literature e.g. in Bansal and Yaron (2004), I discuss its effect at length in Section 3 (especiall Section 3.5). I contrast the cases with $\psi = \psi^* = 0.2$ and $\psi = \psi^* = 2$, and ψ turns out to have a large impact on the potency of imperfect financial integration. For the main application of Section 4, I therefore use $\psi = \psi^* = 0.5$, which allows me to generate a plausible home bias in equity holdings while matching the broad level of the interest rate in this asymmetric context. This lower value goes some way towards the much lower estimates of the elasticity of intertemporal substitution that have been used historically in the earlier literature e.g. in Hall (1988), Campbell (1999).

The home bias in consumption $\alpha = \alpha^* = 0.75$ is consistent with the share of import in the consumption basket of the United States and other countries in recent years. The value is in line with the range of values that have been used in the literature, although slightly lower given the slight increase in world trade in recent decades. In the literature, values as high as $\alpha = 0.9$ or even $\alpha = 0.975$ are sometimes necessary from a quantitative perspective, but this is not the case in the context of this paper where I study the dynamics throughout the state space instead of local neighborhoods of a steady-state. Note that, as α increases further, portfolios and other variables become very non-linear, and the impact of the wealth share is strongly reinforced even in the baseline calibration.

The numéraire basket has a weight of $a = 1 - a = 1/2$ on each good. The value of a has no consequence on quantities and only tilts prices accordingly. I therefore stick to a symmetric numéraire basket to ease interpretation. In extensions of the model with more assets (e.g. multiple bonds), portfolio constraints, and additional sources of risk, the denomination of the numéraire could be of more interest, an aspect that I am planning to explore.

The elasticity of substitution between goods $\theta = \theta^* = 2$ is in line with modern standard estimates e.g. in Imbs and Méjean (2015), and as used in the literature. Cf. among others Tille (2001), Corsetti et al. (2008), Coeurdacier (2009), Obstfeld (2007), Bhamra et al. (2014) for a discussion. I take a value slightly lower than Imbs and Méjean (2015)'s preferred range of $[4, 6]$, as a compromise towards the lower values that had been used in the earlier literature. From an economic standpoint, most relevant is that this elasticity is above one, a point whose impact I discuss at length throughout Section 3, and in particular in Section 3.4 on portfolios.

The discount rate is standard at $\rho = \rho^* = 1\%$, and allows to match the broad level of the interest rate.

In the main text of the paper, labor income is inactive: $\delta = \delta^* = 0\%$. I briefly cover the impact of labor income, which has been discussed in the literature, in Appendix A.8. In that case, I use $\delta = \delta^* = 62.5\%$, in line with the average labor share in the United States over the last 50 years.

The tax on foreign dividends, which captures imperfect financial integration, is set to $\tau = \tau^* = 0\%$ in the baseline. Its effect is discussed at length in Section 3.5, and some more in Section 4.

Output processes have a growth rate in annual terms of $\mu_Y = \mu_{Y*} = 2\%$, and a volatility of $\sigma_{Yz} = \sigma_{Y*z*} = 4.1\%$. This is in line with typical values used in the literature, and broadly consistent with world averages e.g. in Uribe and Schmitt-Grohé (2017), in International Monetary Fund or World Bank data, or in longer-run series in Jordà et al. (2016). Asymmetries in output growth rates and volatilities could be an interesting exploration from the perspective of studying the integration of developed slower-growing countries, with emerging faster-growing economies. Importantly, the fundamental correlation between the output of each tree is assumed to be zero. This is not meant to capture empirical correlations, but allows to focus on the correlation between asset returns and goods prices that emerge purely endogenously.

A.8. Impact of labor income

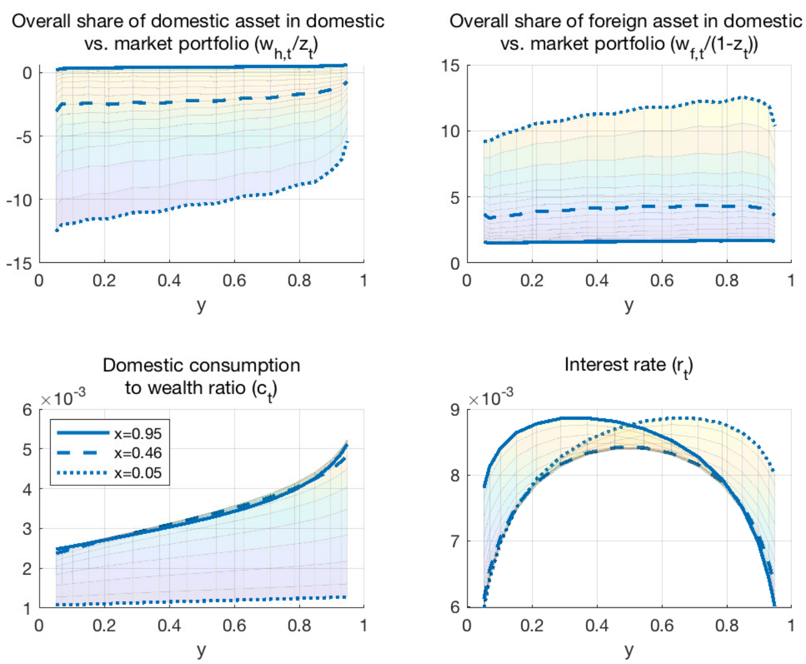
(Back to main text: Section 3.5.)

The heterogeneity of investors is another factor that strongly reinforces the influence of the wealth share on the equilibrium, not only conceptually but also quantitatively. This was already apparent in the analysis of the baseline calibration studied so far. As we have seen, for instance in Table 1, an increase in the home bias in consumption, which constitutes the fundamental heterogeneity in the economy, increases the impact of the wealth share significantly.

Here, I briefly study the impact of heterogeneity further by staying in a symmetric calibration but introducing labor income. Heterogeneity is also partly the focus of the application of Section 4, albeit of a different kind, as the investors will exhibit asymmetries in tolerance for risk.

As a reminder, labor income is modeled as a constant share ($\delta = \delta^*$) of the output of each tree being paid to the local investor. By making the budget constraint of each investor more dependent on the local output, labor income also increases the heterogeneity between investors in the world economy. While its effect on risk premia and Sharpe ratios is somewhat modest, labor income significantly affects portfolios, marginal values of wealth, consumptions, and the interest rate. Those are shown in Figure A.1 for a labor share δ of 62.5%, roughly in line with the average labor share in the United States over the last 50 years. In addition, its effect is once again going hand-in-hand with a bolstered importance for the wealth share.

Figure A.1: Impact of the wealth share in the presence of labor income ($\delta = 62.5\%$)

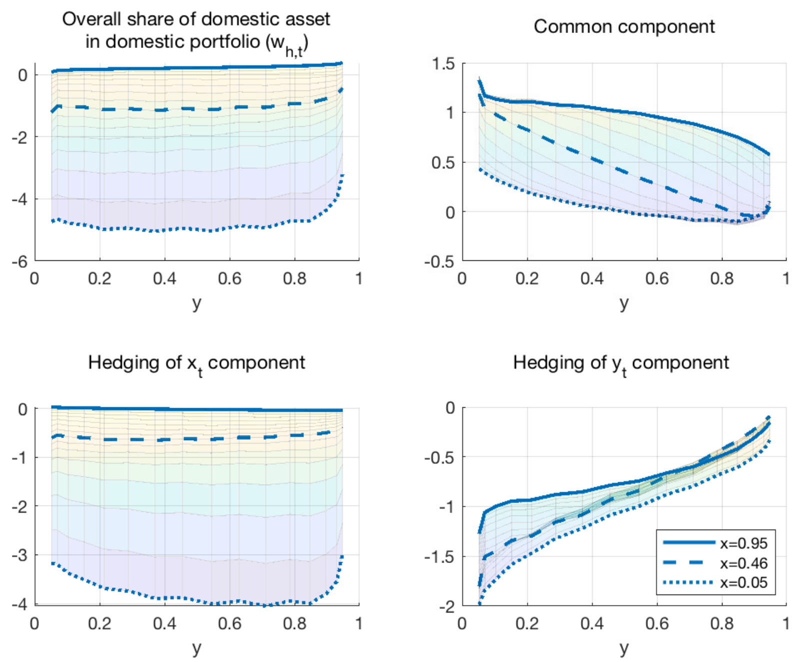


Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except that $\tau = 62.5\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

The top two panels of Figure A.1 show portfolio weights as they compare to the market portfolio, HB_t and FB_t. Because labor income is perfectly correlated with the payoff of the local asset, it renders each asset yet more unattractive to the local investor, therefore reinforcing the foreign bias in equity holdings on average. This is in line with Baxter and Jermann (1997), who argue that "The International Diversification Puzzle Is Worse Than

You Think" when labor takes this form. In terms of magnitude, the impact is substantial, with the measure of home bias now varying from -12.5 to 1 as the wealth share increases, an effect of much larger magnitude than that of fundamentals. In addition, portfolios change not only on average but also inherently in a state-dependent fashion, with the foreign bias reinforced in particular as an investor holds an increasingly smaller share of world wealth. Take the domestic investor for instance: as her wealth share decreases towards zero, labor income represents an increasingly larger share of her revenues, making hedging the labor income risk increasingly important. Due to the perfect correlation between domestic labor income and the payoff to the domestic asset, this pushes the domestic investor to tilt her portfolio away from the domestic asset some more.

Figure A.2: Components of the domestic portfolio in the presence of labor income (δ = 62.5%)



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Portfolios are not only affected in their overall shape, but also in their underlying drivers. This can be observed visually in Figure A.2, which reports the weight of the domestic asset in the domestic portfolio as well as its components, and is confirmed by computing the corresponding variance decomposition of $w_{h,t}$ like before. From both, we observe that the share of $w_{h,t}$ explained by the hedging of x_t increases tremendously, going from 7.1% in the baseline without labor income, to a whooping 70.2% for $\delta = 62.5\%$. On the contrary, the common and y_t-hedging components now explain a mere 19.6% and 10.3%, instead of 69.7% and 23.1% in the baseline. In short: the hedging of wealth share risk becomes the main driver of the shape of portfolios.

Labor income also has a significant impact on marginal values of wealth, and therefore on consumptions, both becoming more dependent on the wealth share than in the baseline in which x_t affected them only modestly. For instance, the marginal value of wealth for the domestic investor decreases more markedly as the wealth share gets smaller, due to the fact that domestic labor income represents an increasing amount in comparison to domestic wealth, ensuring that the domestic investor has comparatively more resources to fund its consumption and portfolios. As a result, while the average level of consumption to wealth is broadly unchanged, domestic consumption significantly decreases as a fraction of wealth when $x_t \to 0$, as shown in the bottom left panel of Figure A.1. Interestingly, this pattern is reversed and domestic consumption increases as $x_t \to 0$, when the elasticity of intertemporal substitution ψ is small, emphasizing the impact of ψ on the relative importance of substitution and income effects.[85] When ψ is large, in particular above 1, the substitution effect is strong so that an investor ends up saving a large part of the extra labor income (as a fraction of wealth), resulting in a lower consumption as a fraction of wealth when their wealth share decreases. Conversely, as ψ is small, in particular below 1, the income effect dominates so that an investor ends up spending most of the extra labor income (as a function of wealth) on increased consumption as their wealth share decreases. This phenomenon points once again to the importance of being able to study these mechanisms in a context with general preferences, solved globally throughout the state space.

The pattern for the interest rate mirrors those for the marginal values of wealth and consumptions.[86] On average, r_t slightly decreases compared to the baseline, by about 21 basis points throughout the state space, reflecting the fact that an addition risk, the labor income, needs to be hedged in this economy[87], but more noticeable is the impact on the shape. The interest rate becomes more asymmetric as a function of relative output, going e.g. from around 0.6% to 0.8% depending on whether $y_t \to 0$ or $y_t \to 1$ when the domestic investor holds a small share of world wealth. This represents a reinforcement of the driver of r_t in the baseline combined with a larger investor heterogeneity. In addition, the evolution of r_t as a function of the wealth share is also worth pointing out: as x_t gets small, the interest rate noticeably increases, which has to happen in equilibrium for the domestic investor to be willing to significantly cut down on consumption. Like before, this pattern is also reversed for small values of the elasticity of intertemporal substitution, with the interest rate *de*creasing

[85] I use the terms "substitution effect" and "income effect" liberally, in contrast to their more usual and restricted use that relates to the impact of the interest rate.

[86] This is also true for the pattern of the domestic and foreign dividend yields, F_t and F_t^*, which appear in the budget constraints once we divide labor income by wealth: $\delta F_t z_t/((1-\delta)x_t)$ for the domestic investor, and $\delta^* F_t^*(1-z_t)/((1-\delta^*)(1-x_t))$ for the foreign investor. Cf. Section 2.4.

[87] This effect is limited because of the perfect correlation between labor income and the payoff of the local asset.

as the wealth share gets close to zero or one in that case.

Lastly, the introduction of labor income has non-linear effects on the equilibrium distribution of state variables, as shown in Figure F.5. While the dispersion of the wealth share first decreases with δ, consistent with labor income tightening the wealth distribution by ensuring a minimum level of revenues for each investor, dispersion increases back for large values of δ. In addition, as δ increases, the steepness of the relationship between x_t and y_t increases. Those effects are the results of the interplay between the several components of the drift and diffusion of the wealth share, shown in Figure XX. Note also that the second effect, with dispersion increasing back with δ, tends to occur faster for lower level of the elasticity of intertemporal substitution ψ.

Overall, labor income has a significant impact on the equilibrium and its underpinnings due to the resulting increased heterogeneity that reinforces the impact of the wealth share. The way those patterns change when considering a more general and realistic specification for labor income could prove an interesting exploration. One particular specification could be to construct labor income as a time-varying share of the output of each country, as explored for instance in Coeurdacier and Gourinchas (2016). As the authors suggest, the correlation of labor income with output, once computed with the proper conditioning, could in fact turn out to be negative, providing a natural way to generate a home bias in equity holdings. If the share is itself stochastic, it could also provide an additional hedging motive that could prove relevant in practice also as it introduces a natural degree of market incompleteness. Labor income could also take a more general form, for instance as a separate source of idiosyncratic risk in the spirit of the recent heterogeneous-agent macroeconomic literature like Kaplan et al. (2018), or by introducing a distribution of investors in each country by generalizing the overlapping generation structure of Gârleanu and Panageas (2015) to a two-good, two-country setting. The latter hints at how labor income could help both (types of) investors survive in equilibrium.[88] I leave these promising avenues for future research.

[88] One difficulty is that this might generate a stationary distribution between investors within a country, as a constant share of them is assumed so switch between different groups of investors, but it would not be sufficient *per se* to ensure a stationary distribution of wealth between *international* investors, except by assuming that individual investors can switch between countries. The ability of labor income to ensure the survival of different types of agents is also used in He and Krishnamurthy (2013).

B. Proofs

The proof that the equilibrium can be recast as a stationary recursive Markovian equilibrium with $X = (x, y)'$ as state variables follows a guess and verify approach, e.g. as in Gârleanu and Panageas (2015).

B.1. HJBs and Propositions

Following the usual argument, (3) and (A.8) can be reformulated as the following Hamilton-Jacobi-Bellman equations, subject to the same budget constraints and goods aggregators

$$0 = \max_{C_t, w_{h,t}, w_{f,t}} f(C_t, V_t)dt + \mathbb{E}_t [dV_t] \tag{B.1}$$
$$\text{subject to (4) \& (5)}$$

$$0 = \max_{C_t^*, w_{h,t}^*, w_{f,t}^*} f(C_t^*, V_t^*)dt + \mathbb{E}_t [dV_t^*] \tag{B.2}$$
$$\text{subject to (A.10) \& (A.11)}$$

Using the homotheticity of the value function with recursive preferences, one can show that

$$V(W, x, y) = \left(\frac{W^{1-\gamma}}{1-\gamma}\right) J(x, y)^{\frac{1-\gamma}{1-\psi}} \tag{B.3}$$

$$V^*(W^*, x, y) = \left(\frac{W^{*1-\gamma^*}}{1-\gamma^*}\right) J^*(x, y)^{\frac{1-\gamma^*}{1-\psi^*}} \tag{B.4}$$

where $J_t = J(x_t, y_t), J_t^* = J^*(x_t, y_t)$ are two unknown functions to solve for. For CRRA utility, the expressions simplify to

$$V(W, x, y) = \left(\frac{W^{1-\gamma}}{1-\gamma}\right) J(x, y)^{-\gamma} \tag{B.5}$$

while for log utility, they simplify to

$$V(W, x, y) = \frac{1}{\rho} \log W + J(x, y) \tag{B.6}$$

Using Itô's Lemma to compute dV_t and simplifying, we obtain the following Hamilton-Jacobi-Bellman equation for the home country

$$\begin{aligned}
0 = \max_{c_t, w_{h,t}, w_{f,t}} & \left(\frac{1}{1-1/\psi}\right)\left[\left(\frac{c_t}{J_t^{1/(1-\psi)}}\right)^{1-1/\psi} - \rho\right] \\
& + \left(r_t + w_{h,t}\left(\mu_{R,t} - r_t\right) + w_{f,t}\left(\mu_{R*,t} - r_t\right) - P_t c_t\right) \\
& + \left(\frac{1}{1-\psi}\right)\mu_{J,t} \\
& - \frac{\gamma}{2}\left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right)^T\left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right) \\
& + \frac{1}{2}\left(\frac{1}{1-\psi}\right)\left(\frac{\psi-\gamma}{1-\psi}\right)\sigma_{J,t}^T\sigma_{J,t} \\
& + \left(\frac{1-\gamma}{1-\psi}\right)\left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right)^T\sigma_{J,t}
\end{aligned} \tag{B.7}$$

For the foreign country, the Hamilton-Jacobi-Bellman equation is

$$\begin{aligned}
0 = \max_{c_t^*, w_{h,t}^*, w_{f,t}^*} & \left(\frac{1}{1-1/\psi^*}\right)\left[\left(\frac{c_t^*}{J_t^{*1/(1-\psi^*)}}\right)^{1-1/\psi^*} - \rho^*\right] \\
& + \left(r_t + w_{h,t}^*\left(\mu_{R,t} - r_t\right) + w_{f,t}^*\left(\mu_{R*,t} - r_t\right) - P_t^* c_t^*\right) \\
& + \left(\frac{1}{1-\psi^*}\right)\mu_{J*,t} \\
& - \frac{\gamma^*}{2}\left(w_{h,t}^*\sigma_{R,t} + w_{f,t}^*\sigma_{R*,t}\right)^T\left(w_{h,t}^*\sigma_{R,t} + w_{f,t}^*\sigma_{R*,t}\right) \\
& + \frac{1}{2}\left(\frac{1}{1-\psi^*}\right)\left(\frac{\psi^*-\gamma^*}{1-\psi^*}\right)\sigma_{J*,t}^T\sigma_{J*,t} \\
& + \left(\frac{1-\gamma^*}{1-\psi^*}\right)\left(w_{h,t}^*\sigma_{R,t} + w_{f,t}^*\sigma_{R*,t}\right)^T\sigma_{J*,t}
\end{aligned} \tag{B.8}$$

where following Remark A.1

$$\frac{dJ_t}{J_t} \equiv \mu_{J,t}dt + \sigma_{J,t}^T d\vec{z}_t \tag{B.9}$$

$$\mu_{J,t} = \frac{J_{x,t}}{J_t}x_t\mu_{x,t} + \frac{J_{y,t}}{J_t}y_t\mu_{y,t} + \frac{1}{2}\frac{J_{xx,t}}{J_t}x_t^2\sigma_{x,t}^T\sigma_{x,t} + \frac{1}{2}\frac{J_{yy,t}}{J_t}y_t^2\sigma_{y,t}^T\sigma_{y,t} + \frac{J_{xy,t}}{J_t}x_t y_t\sigma_{x,t}^T\sigma_{y,t} \tag{B.10}$$

$$\sigma_{J,t} = \frac{J_{x,t}}{J_t}x_t\sigma_{x,t} + \frac{J_{y,t}}{J_t}y_t\sigma_{y,t} \tag{B.11}$$

and

$$\frac{dJ_t^*}{J_t^*} \equiv \mu_{J*,t}dt + \sigma_{J*,t}^T d\vec{z}_t \tag{B.12}$$

$$\mu_{J*,t} = \left(\frac{J_{x,t}^*}{J_t^*}x_t\mu_{x,t} + \frac{J_{y,t}^*}{J_t^*}y_t\mu_{y,t} + \frac{1}{2}\frac{J_{xx,t}^*}{J_t^*}x_t^2\sigma_{x,t}^T\sigma_{x,t} + \frac{1}{2}\frac{J_{yy,t}^*}{J_t^*}y_t^2\sigma_{y,t}^T\sigma_{y,t} + \frac{J_{xy,t}^*}{J_t^*}x_t y_t\sigma_{x,t}^T\sigma_{y,t}\right) \tag{B.13}$$

$$\sigma_{J*,t} = \frac{J_{x,t}^*}{J_t^*}x_t\sigma_{x,t} + \frac{J_{y,t}^*}{J_t^*}y_t\sigma_{y,t} \tag{B.14}$$

Taking first-order conditions with respect to $c_t, w_{h,t}, w_{f,t}$ and $c_t^*, w_{h,t}^*, w_{f,t}^*$, respectively, yields Propositions 5 and A.3. Plugging back in the equations above delivers the Hamilton-Jacobi-Bellman equations in Proposition A.2. Prices in Propositions 2 and A.4, and risk premia in Proposition 4, are obtained by combining those expressions with the several market-clearing conditions for goods and assets.

B.2. Stochastic discount factors

The stochastic discount factors of the domestic and foreign investors are

$$\xi_t \equiv \xi_0 \exp\left\{\int_0^t \frac{\partial f}{\partial V}(C_u, V_u)\,du\right\}\frac{\partial V_t}{\partial W_t} = \exp\left\{\int_0^t \frac{\partial f}{\partial V}(C_u, V_u)\,du\right\}W_t^{-\gamma}J_t^{\frac{1-\gamma}{1-\psi}} \tag{B.15}$$

$$\xi_t^* \equiv \xi_0^* \exp\left\{\int_0^t \frac{\partial f^*}{\partial V^*}(C_u^*, V_u^*)\,du\right\}\frac{\partial V_t^*}{\partial W_t^*} = \exp\left\{\int_0^t \frac{\partial f^*}{\partial V^*}(C_u^*, V_u^*)\,du\right\}W_t^{*-\gamma^*}J_t^{*\frac{1-\gamma^*}{1-\psi^*}} \tag{B.16}$$

Let us focus on the home investor. The foreign investor is similar. It follows that

$$\ln \xi_t = \int_0^t \frac{\partial f}{\partial V}(C_u, V_u)\,du + \left(\frac{1-\gamma}{1-\psi}\right)\ln J_t - \gamma \ln W_t \tag{B.17}$$

$$\Rightarrow d\ln \xi_t = \frac{\partial f}{\partial V}(C_t, V_t)\,dt + \left(\frac{1-\gamma}{1-\psi}\right)d\ln J_t - \gamma d\ln W_t \equiv \mu_{\ln \xi,t}dt + \sigma_{\ln \xi,t}^T d\vec{Z}_t \tag{B.18}$$

From the definition of $f(C, V)$ in Equation (3), one can show that (algebra or cf. e.g. Gârleanu and Panageas (2015), Duffie and Epstein, 1992, Schroder and Skiadas (1999)):

$$\frac{\partial f}{\partial V}(C_t, V_t)\,dt = \Theta_1 P_t^{1-\psi}J_t + \Theta_2 \tag{B.19}$$

with constants

$$\Theta_1 \equiv -\left(\frac{\gamma - \frac{1}{\psi}}{1-\frac{1}{\psi}}\right) \quad \text{and} \quad \Theta_2 \equiv \frac{\rho(\gamma-1)}{1-\frac{1}{\psi}}$$

In addition

$$d\ln J_t \equiv \mu_{\ln J,t}dt + \sigma_{\ln J,t}^T d\vec{Z}_t = \left(\mu_{J,t} - \frac{1}{2}\sigma_{J,t}^T\sigma_{J,t}\right)dt + \sigma_{J,t}^T d\vec{Z}_t \tag{B.20}$$

$$d\ln W_t \equiv \mu_{\ln W,t}dt + \sigma_{\ln W,t}^T d\vec{Z}_t = \left(\mu_{W,t} - \frac{1}{2}\sigma_{W,t}^T\sigma_{W,t}\right)dt + \sigma_{W,t}^T d\vec{Z}_t \tag{B.21}$$

$$\frac{dJ_t}{J_t} \equiv \mu_{J,t}dt + \sigma_{J,t}^T d\vec{Z}_t \tag{B.22}$$

$$\mu_{J,t} \equiv \frac{J_{x,t}}{J_t}x_t\mu_{x,t} + \frac{J_{y,t}}{J_t}y_t\mu_{y,t} + \frac{1}{2}\frac{J_{xx,t}}{J_t}x_t^2\sigma_{x,t}^T\sigma_{x,t} + \frac{1}{2}\frac{J_{yy,t}}{J_t}y_t^2\sigma_{y,t}^T\sigma_{y,t} + \frac{J_{xy,t}}{J_t}x_t y_t\sigma_{x,t}^T\sigma_{y,t}$$

$$\sigma_{J,t} \equiv \frac{J_{x,t}}{J_t}x_t\sigma_{x,t} + \frac{J_{y,t}}{J_t}y_t\sigma_{y,t}$$

and $\mu_{W,t}, \sigma_{W,t}$ are given in Equation (5) repeated here for convenience:

$$\frac{dW_t}{W_t} = \left(r_t + w_{h,t}\left(\mu_{R,t} - r_t\right) + w_{f,t}\left(\mu_{R*,t} - r_t\right) - P_t c_t\right)dt$$
$$+ \left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right)^T d\vec{Z}_t$$

Therefore:

$$\mu_{\ln\xi,t} = \Theta_1 P_t^{1-\psi}J_t + \Theta_2 + \left(\frac{1-\gamma}{1-\psi}\right)\left(\mu_{J,t} - \frac{1}{2}\sigma_{J,t}^T\sigma_{J,t}\right) - \gamma\left(\mu_{W,t} - \frac{1}{2}\sigma_{W,t}^T\sigma_{W,t}\right) \tag{B.23}$$

$$\sigma_{\ln\xi,t} = \left(\frac{1-\gamma}{1-\psi}\right)\sigma_{J,t} - \gamma\sigma_{W,t} \tag{B.24}$$

Finally:

$$\frac{d\xi_t}{\xi_t} \equiv \mu_{\xi,t}dt + \sigma_{\xi,t}^T d\vec{Z}_t = \left(\mu_{\ln\xi,t} + \frac{1}{2}\sigma_{\ln\xi,t}^T\sigma_{\ln\xi,t}\right)dt + \sigma_{\ln\xi,t}^T d\vec{Z}_t \tag{B.25}$$

Plugging all components, we obtain:

$$
\begin{aligned}
\mu_{\xi,t} &= \mu_{\ln\xi,t} + \frac{1}{2}\sigma_{\ln\xi,t}^T \sigma_{\ln\xi,t} \qquad \text{(B.26)} \\
&= \Theta_1 P_t^{1-\psi} J_t + \Theta_2 + \left(\frac{1-\gamma}{1-\psi}\right)\left(\mu_{J,t} - \frac{1}{2}\sigma_{J,t}^T \sigma_{J,t}\right) - \gamma\left(\mu_{W,t} - \frac{1}{2}\sigma_{W,t}^T \sigma_{W,t}\right) \\
&\quad + \frac{1}{2}\left(\left(\frac{1-\gamma}{1-\psi}\right)\sigma_{J,t} - \gamma\sigma_{W,t}\right)^T \left(\left(\frac{1-\gamma}{1-\psi}\right)\sigma_{J,t} - \gamma\sigma_{W,t}\right) \\
&= \Theta_1 P_t^{1-\psi} J_t + \Theta_2 + \left(\frac{1-\gamma}{1-\psi}\right)\left(\mu_{J,t} - \frac{1}{2}\sigma_{J,t}^T \sigma_{J,t}\right) \\
&\quad - \gamma\left(r_t + w_{h,t}\left(\mu_{R,t} - r_t\right) + w_{f,t}\left(\mu_{R*,t} - r_t\right) - P_t^{1-\psi} J_t\right) \\
&\quad + \frac{\gamma}{2}\left(\left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right)^T \left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right)\right) \\
&\quad + \frac{1}{2}\left(\left(\frac{1-\gamma}{1-\psi}\right)\sigma_{J,t} - \gamma\left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right)\right)^T \\
&\qquad \left(\left(\frac{1-\gamma}{1-\psi}\right)\sigma_{J,t} - \gamma\left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right)\right) \\
\sigma_{\xi,t} &= \left(\frac{1-\gamma}{1-\psi}\right)\sigma_{J,t} - \gamma\left(w_{h,t}\sigma_{R,t} + w_{f,t}\sigma_{R*,t}\right) \qquad \text{(B.27)}
\end{aligned}
$$

Similarly, for the foreign investor:

$$
\begin{aligned}
\mu_{\xi*,t} &= \Theta_1^* P_t^{*1-\psi^*} J_t^* + \Theta_2^* + \left(\frac{1-\gamma^*}{1-\psi^*}\right)\left(\mu_{J*,t} - \frac{1}{2}\sigma_{J*,t}^T \sigma_{J*,t}\right) \qquad \text{(B.28)} \\
&\quad - \gamma\left(r_t + w_{h,t}^*\left(\mu_{R,t} - r_t\right) + w_{f,t}^*\left(\mu_{R*,t} - r_t\right) - P_t^{*1-\psi} J_t^*\right) \\
&\quad + \frac{\gamma}{2}\left(\left(w_{h,t}^*\sigma_{R,t} + w_{f,t}^*\sigma_{R*,t}\right)^T \left(w_{h,t}^*\sigma_{R,t} + w_{f,t}^*\sigma_{R*,t}\right)\right) \\
&\quad + \frac{1}{2}\left(\left(\frac{1-\gamma^*}{1-\psi^*}\right)\sigma_{J*,t} - \gamma^*\left(w_{h,t}^*\sigma_{R,t} + w_{f,t}^*\sigma_{R*,t}\right)\right)^T \\
&\qquad \left(\left(\frac{1-\gamma^*}{1-\psi^*}\right)\sigma_{J*,t} - \gamma^*\left(w_{h,t}^*\sigma_{R,t} + w_{f,t}^*\sigma_{R*,t}\right)\right) \\
\sigma_{\xi*,t} &= \left(\frac{1-\gamma^*}{1-\psi^*}\right)\sigma_{J*,t} - \gamma^*\left(w_{h,t}^*\sigma_{R,t} + w_{f,t}^*\sigma_{R*,t}\right) \qquad \text{(B.29)}
\end{aligned}
$$

Under complete markets:

$$
\begin{aligned}
\frac{d\xi_t}{\xi_t} &= -r_t dt - \kappa_t^T d\vec{Z}_t \qquad \text{(B.30)} \\
\frac{d\xi_t^*}{\xi_t^*} &= -r_t dt - \kappa_t^T d\vec{Z}_t
\end{aligned}
$$

where r_t, κ_t are the interest rate and the (two-dimensional) price of risk that are equal for both investors when markets are complete and risk sharing is perfect.

One can show that:

$$\begin{aligned}\kappa_t &= \gamma_t\{z_t\sigma_{R,t} + (1 - z_t)\sigma_{R^*,t}\} \\ &- \gamma_t \left\{ x_t \left(\frac{1}{\gamma}\right) \left(\frac{1-\gamma}{1-\psi}\right) \sigma_{J,t} + (1 - x_t) \left(\frac{1}{\gamma^*}\right) \left(\frac{1-\gamma^*}{1-\psi^*}\right) \sigma_{J^*,t} \right\}\end{aligned} \tag{B.31}$$

where, as a reminder, $z_t = \frac{Q_t}{Q_t+Q_t^*} = \frac{Q_t}{W_t+W_t^*}$ is the ratio of domestic equity price to world wealth.

Next: expression for r_t and special cases.

C. Numerical Resolution

To solve the problem numerically, I use projection methods based on Chebyshev polynomials and orthogonal collocation. The method follows Judd (1992, 1998). It is also presented in the NBER Summer SI Lecture by Fernández-Villaverde and Christiano (2011) and in Parra-Alvarez (2018) among others, and is applied to multi-agent asset pricing models for instance in Drechsler et al. (2018), Fang (2019), and Kargar (2019).

The model can be written as a system of equations

$$\mathcal{H}(\mathbf{G}) = \mathbf{0} \tag{C.1}$$

where $\mathbf{G} : [0,1] \times [0,1] \rightarrow \mathbb{R}^M$ is a function of the state variables $X = (x, y)'$: $\mathbf{G}(X)$. $\mathcal{H} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is an operator, where $\mathcal{B}_1, \mathcal{B}_2$ are spaces of functions. $\mathbf{0}$ is the zero of $\mathcal{B}_2$.

The name of the game is to solve an approximate version of (C.1)

$$\hat{\mathcal{H}}(\hat{\mathbf{G}}) \approx \mathbf{0} \qquad (\mathcal{H}(\hat{\mathbf{G}}) \approx \mathbf{0} \text{ in our case}) \tag{C.2}$$

Specifically, I pick a basis $\{\Psi_{ij}(x,y)\}_{i=1,j=1}^{N,N}$ for the space of functions and use it to approximate the following variables: $\mathcal{G} \equiv \{J_t,\ J_t^*,\ F_t,\ F_t^*,\ q_t,\ w_{h,t},\ w_{f,t}\}$. All other variables and quantities of the model can be expressed as a function of those variables.

Any $g : [0,1] \times [0,1] \rightarrow D^g \subset \mathbb{R}$ in $\mathcal{G}$ is approximated at the order N as follows

$$\hat{g}(X) = \sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij}^{(N)} \Psi_{ij}^{(N)}(x, y) \tag{C.3}$$

where $a_{ij}^{(N)}$ are coefficients to solve for.

I use the tensor product of Chebyshev polynomials of order 0 to N as basis

$$\Psi_{ij}^{(N)}(X) = T_i\left(\omega(x)\right) T_i\left(\omega(y)\right) \tag{C.4}$$

where $\omega(x) = 2(x-1), \omega(y) = 2(y-1)$ transform x and y from $[0,1]$ to $[-1,1]$ over which Chebyshev polynomials are defined.

Define the residual function as

$$\mathcal{R}(X; \mathbf{a}) \equiv \hat{\mathcal{H}}(\hat{\mathbf{G}}(X)) \tag{C.5}$$

Once each variable is expressed as a function of $g \in \mathcal{G}$ and state variables $X = (x, y)'$, $\hat{\mathbf{G}}$ and $\mathcal{R}(X; \mathbf{a})$ can be constructed. The last (and main) step is to find the vector of coefficients $\mathbf{a}$ so that

$$\mathcal{R}(X; \mathbf{a}) \approx \mathbf{0} \tag{C.6}$$

More precisely, for some objective function ρ, I pick

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} \rho\left(\mathcal{R}(X; \mathbf{a}), \mathbf{0}\right) \tag{C.7}$$

There exist different methods depending on the choice of ρ (i.e. different ways to project): weighted least squares, Galerkin methods, method of moments, or collocation methods. For the latter, the weight function is the Dirac delta function, i.e. the residual is set to 0 at specific points of the state space. For the orthogonal collocation that I use here, the collocation points are picked as the zeros of the basis, i.e. the Chebyshev zeros. In practice, I use $N = 30$ in most cases, and build the basis using the *CompEcon* package of Miranda and Fackler (2004). The optimization is based on the *fsolve* function of Matlab, and is checked with a number of optimizers from the Global Optimization Toolbox.

Instead of using the tensor product, refined ways of constructing the basis and grid are also possible such as complete polynomials or Smolyak's algorithm. They are not necessary here but could prove useful when the number of state variables increases. The approximation can also be based on a number of other polynomials such as splines.

However, for high-dimensional settings such as the ones likely to arise for extensions of the framework in this paper, those methods rapidly become computationally too costly. This is particularly so if the order of approximation needs to be high due to the presence of strong non-linearities (e.g. with the introduction of portfolio constraints). An alternative that seems to have promise in that context is to extend projection methods by replacing the Chebysev approximation by a neural network approximation, which is naturally able to handle high-dimensional cases. I am developing those "projection methods via neural networks" for continuous-time models in Sauzet (2020b). Details are provided in Appendix E.3.

D. Special Cases

D.1. Planner under symmetric CRRA preferences

The social planner problem under CRRA and symmetric preferences (same parameters, except for home bias in consumption α, which is symmetric) is as follows

$$\max_{\{C_{h,u},C_{f,u},C^*_{h,u},C^*_{f,u}\}_u^\infty} \mathbb{E}_t \left[\int_t^\infty e^{-\rho(u-t)} \left(\lambda \frac{C_u^{1-\gamma}}{1-\gamma} + (1-\lambda)\frac{C_u^{*1-\gamma}}{1-\gamma}\right) du\right] \tag{D.1}$$

subject to

$$dY_u = \mu_{Y,u} du + \sigma^T_{Y,u} d\vec{z}_u \tag{D.2}$$

$$dY^*_u = \mu_{Y*,u} du + \sigma^T_{Y*,u} d\vec{z}_u \tag{D.3}$$

$$C_{h,u} + C^*_{h,u} = Y_u \tag{D.4}$$

$$C_{f,u} + C^*_{f,u} = Y^*_u \tag{D.5}$$

$$C_u = \left[\alpha^{\frac{1}{\theta}} C_{h,u}^{\frac{\theta-1}{\theta}} + (1-\alpha)^{\frac{1}{\theta}} C_{f,u}^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{D.6}$$

$$C^*_u = \left[(1-\alpha)^{\frac{1}{\theta}} C_{h,u}^{*\frac{\theta-1}{\theta}} + \alpha^{\frac{1}{\theta}} C_{f,u}^{*\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{D.7}$$

Plugging the market-clearing condition for the two goods, and taking first-order conditions with respect to the home consumption of each gives

$$\lambda\alpha^{\frac{1}{\theta}} C_t^{\frac{1}{\theta}-\gamma} C_{h,t}^{-\frac{1}{\theta}} = (1-\lambda)(1-\alpha)^{\frac{1}{\theta}} C_t^{*\frac{1}{\theta}-\gamma} C_{h,t}^{*-\frac{1}{\theta}} \tag{D.8}$$

$$\lambda(1-\alpha)^{\frac{1}{\theta}} C_t^{\frac{1}{\theta}-\gamma} C_{f,t}^{-\frac{1}{\theta}} = (1-\lambda)\alpha^{\frac{1}{\theta}} C_t^{*\frac{1}{\theta}-\gamma} C_{f,t}^{*-\frac{1}{\theta}} \tag{D.9}$$

Reorganizing:

$$\frac{C^*_{h,t}}{C_{h,t}} = \left(\frac{1-\lambda}{\lambda}\right)^\theta \left(\frac{1-\alpha}{\alpha}\right) \left(\frac{C^*_t}{C_t}\right)^{1-\gamma\theta} \tag{D.10}$$

$$\frac{C^*_{f,t}}{C_{f,t}} = \left(\frac{1-\lambda}{\lambda}\right)^\theta \left(\frac{\alpha}{1-\alpha}\right) \left(\frac{C^*_t}{C_t}\right)^{1-\gamma\theta} \tag{D.11}$$

Also note that:

$$\frac{C^*_{f,t}}{C^*_{h,t}} = \left(\frac{\alpha}{1-\alpha}\right)^2 \frac{C_{f,t}}{C_{h,t}} \tag{D.12}$$

Let us use a detour via the decentralized problem and prices to make progress easily. From the static optimization for consumption baskets:

$$C_{h,t} = \alpha \left(\frac{p_t}{P_t}\right)^{-\theta} C_t \tag{D.13}$$

$$C_{f,t} = (1-\alpha) \left(\frac{p_t^*}{P_t}\right)^{-\theta} C_t \tag{D.14}$$

$$C_{h,t}^* = (1-\alpha) \left(\frac{p_t}{P_t^*}\right)^{-\theta} C_t^* \tag{D.15}$$

$$C_{f,t}^* = \alpha \left(\frac{p_t^*}{P_t^*}\right)^{-\theta} C_t^* \tag{D.16}$$

where p_t, p_t^* are prices of goods, and P_t, P_t^* are the prices of the home and foreign consumption basket:

$$P_t = \left[\alpha p_t^{1-\theta} + (1-\alpha) p_t^{*1-\theta}\right]^{\frac{1}{1-\theta}} \tag{D.17}$$

$$P_t^* = \left[(1-\alpha) p_t^{1-\theta} + \alpha p_t^{*1-\theta}\right]^{\frac{1}{1-\theta}} \tag{D.18}$$

Plugging (D.13) and (D.14) in (D.10) yields a relationship between C_t^*/C_t and the real exchange rate $\mathcal{E}_t$

$$\mathcal{E}_t \equiv \frac{P_t^*}{P_t} = \left(\frac{1-\lambda}{\lambda}\right) \left(\frac{C_t^*}{C_t}\right)^{-\gamma} \equiv \phi \left(\frac{C_t^*}{C_t}\right)^{-\gamma} \tag{D.19}$$

$$\Leftrightarrow \frac{C_t^*}{C_t} = \phi^{\frac{1}{\gamma}} \mathcal{E}_t^{-\frac{1}{\gamma}} \tag{D.20}$$

This is nothing but the Backus-Smith condition in this special case. Let us show that $\mathcal{E}_t$ is a function of Y_t/Y_t^* only, so that C_t^*/C_t is too. To do so, I first look for an equation for $q_t \equiv p_t^*/p_t$, the terms of trade, as a function of which $\mathcal{E}_t$ and all other prices can be expressed.

Plugging (D.13) into (D.16) in the market-clearing condition for goods yields

$$\alpha \left(\frac{p_t}{P_t}\right)^{-\theta} C_t + (1-\alpha) \left(\frac{p_t}{P_t^*}\right)^{-\theta} C_t^* = Y_t \tag{D.21}$$

$$(1-\alpha) \left(\frac{p_t^*}{P_t}\right)^{-\theta} C_t + \alpha \left(\frac{p_t^*}{P_t^*}\right)^{-\theta} C_t^* = Y_t^* \tag{D.22}$$

Dividing the two:

$$q_t^{-\theta}\left(\frac{\alpha+(1-\alpha)\mathcal{E}_t^{\theta}\frac{C_t^*}{C_t}}{(1-\alpha)+\alpha\mathcal{E}_t^{\theta}\frac{C_t^*}{C_t}}\right)=\frac{Y_t}{Y_t^*}=\frac{y_t}{1-y_t} \tag{D.23}$$

$$\Rightarrow\ q_t=S_t^{\frac{1}{\theta}}\left(\frac{Y_t}{Y_t^*}\right)^{\frac{1}{\theta}}=S_t^{\frac{1}{\theta}}\left(\frac{y_t}{1-y_t}\right)^{\frac{1}{\theta}} \tag{D.24}$$

where

$$S_t=\frac{(1-\alpha)+\alpha\mathcal{E}_t^{\theta}\frac{C_t^*}{C_t}}{\alpha+(1-\alpha)\mathcal{E}_t^{\theta}\frac{C_t^*}{C_t}}=\frac{(1-\alpha)+\alpha\phi^{\frac{1}{\gamma}}\mathcal{E}_t^{\theta-\frac{1}{\gamma}}}{\alpha+(1-\alpha)\phi^{\frac{1}{\gamma}}\mathcal{E}_t^{\theta-\frac{1}{\gamma}}} \tag{D.25}$$

As a side note, if the IES is equal to the elasticity of substitution between goods ($\psi=\gamma^{-1}=\theta$)

$$q_t=\bar{S}^{\frac{1}{\theta}}\left(\frac{Y_t}{Y_t^*}\right)^{\frac{1}{\theta}}\quad\text{with}\quad\bar{S}=\frac{(1-\alpha)+\alpha\phi^{\frac{1}{\gamma}}}{\alpha+(1-\alpha)\phi^{\frac{1}{\gamma}}} \tag{D.26}$$

To find an equation for q_t in the general case, let us use the expression for $\mathcal{E}_t$ as a function of q_t

$$\mathcal{E}_t=\frac{P_t^*}{P_t}=\left(\frac{(1-\alpha)+\alpha q_t^{1-\theta}}{\alpha+(1-\alpha)q_t^{1-\theta}}\right)^{\frac{1}{1-\theta}} \tag{D.27}$$

Plugging this expression in the above, this yields a non-linear equation for q_t as a function of $Y_t/Y_t^*=y_t/(1-y_t)$

$$q_t^{\theta}=\frac{(1-\alpha)+\alpha\phi^{\frac{1}{\gamma}}\left(\frac{(1-\alpha)+\alpha q_t^{1-\theta}}{\alpha+(1-\alpha)q_t^{1-\theta}}\right)^{\frac{\gamma\theta-1}{\gamma(1-\theta)}}}{\alpha+(1-\alpha)\phi^{\frac{1}{\gamma}}\left(\frac{(1-\alpha)+\alpha q_t^{1-\theta}}{\alpha+(1-\alpha)q_t^{1-\theta}}\right)^{\frac{\gamma\theta-1}{\gamma(1-\theta)}}}\left(\frac{y_t}{1-y_t}\right) \tag{D.28}$$

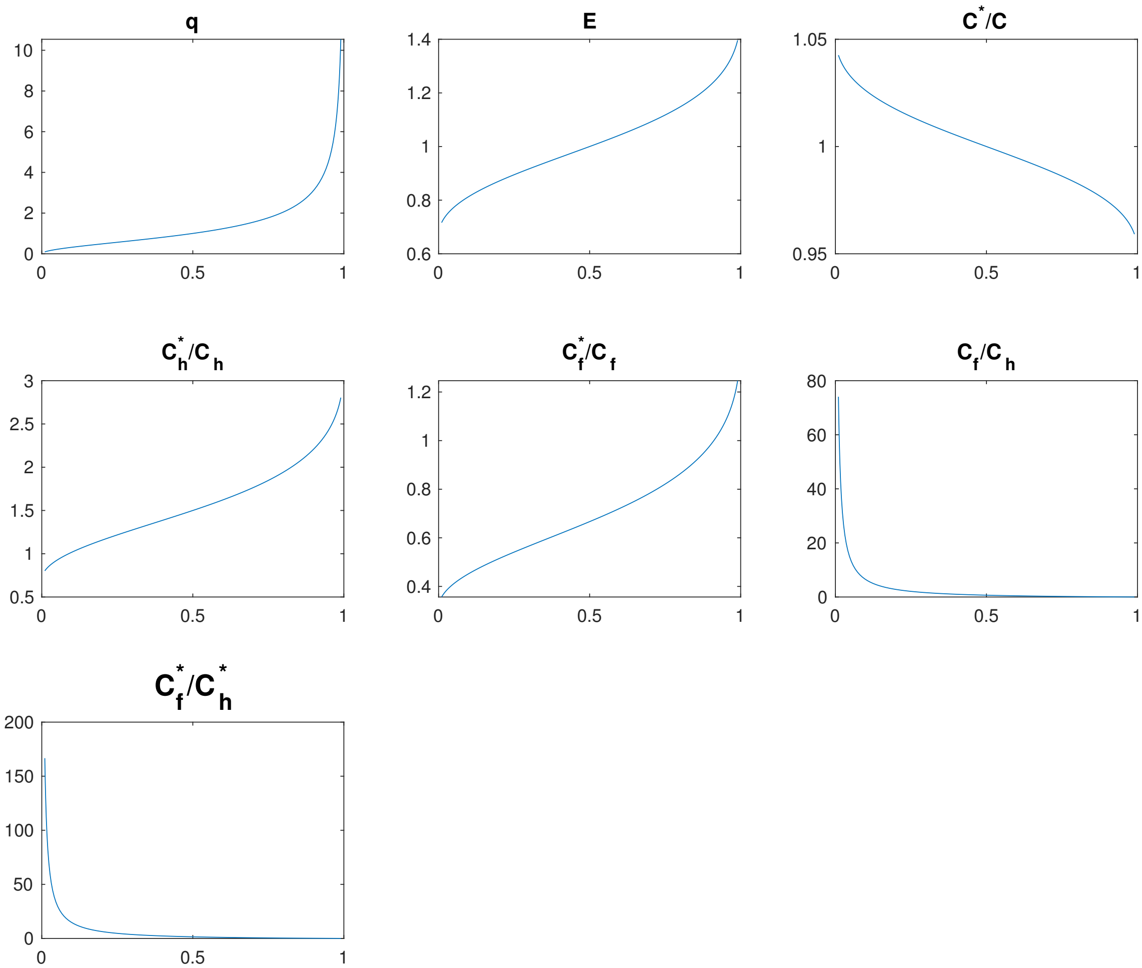
I solve for q_t as a function of $y_t=Y_t/(Y_t+Y_t^*)$ because this variable is in $[0,1]$. This is more stable than to solve for a function on $[0,\infty)$. It also makes comparing this solution to the decentralized one easier. To do so, I approximate $q(y_t)$ using Chebyshev polynomials of order $N=100$, on $N+1$ grid points.

Once I obtain $q_t = q(y_t)$, $\mathcal{E}_t$ follows from (D.27), C_t^*/C_t follows from (D.20), $C_{h,t}^*/C_{h,t}$ and $C_{f,t}^*/C_{f,t}$ from (D.10) and (D.11), and $C_{f,t}/C_{h,t}$ and $C_{f,t}^*/C_{h,t}^*$ from

$$\frac{C_{f,t}}{C_{h,t}} = \frac{(1-\alpha)\left(\frac{p_t^*}{P_t}\right)^{-\theta} C_t}{\alpha\left(\frac{p_t}{P_t}\right)^{-\theta} C_t} = \left(\frac{1-\alpha}{\alpha}\right) q_t^{-\theta} \tag{D.29}$$

$$\frac{C_{f,t}^*}{C_{h,t}^*} = \frac{\alpha\,(p_t^*/P_t^*)^{-\theta}\, C_t^*}{(1-\alpha)\left(\frac{p_t}{P_t^*}\right)^{-\theta} C_t^*} = \left(\frac{\alpha}{1-\alpha}\right) q_t^{-\theta} \tag{D.30}$$

The resulting functions are shown in the Figure below.



To obtain the variables in levels, we can also used the formulas derived above. Denote

$$\frac{C^*_{h,t}}{C_{h,t}} \equiv g_h(y_t) \tag{D.31}$$

$$\frac{C^*_{f,t}}{C_{f,t}} \equiv g_f(y_t) \tag{D.32}$$

Using the market-clearing condition:

$$C_{h,t} = \left(\frac{1}{1+g_h(y_t)}\right) Y_t \equiv h_h(y_t) Y_t \tag{D.33}$$

$$C^*_{h,t} = \left(\frac{g_h(y_t)}{1+g_h(y_t)}\right) Y_t \equiv (1-h_h(y_t)) Y_t \tag{D.34}$$

$$C_{f,t} = \left(\frac{1}{1+g_f(y_t)}\right) Y^*_t \equiv (1-h_f(y_t)) Y^*_t \tag{D.35}$$

$$C^*_{f,t} = \left(\frac{g_f(y_t)}{1+g_f(y_t)}\right) Y^*_t \equiv h_f(y_t) Y^*_t \tag{D.36}$$

Aggregate consumptions can be obtained by plugging the above in their definitions

$$C_t = \left[\alpha^{\frac{1}{\theta}} \left(h_h(y_t) Y_t\right)^{\frac{\theta-1}{\theta}} + (1-\alpha)^{\frac{1}{\theta}} \left(\left(1-h_f(y_t)\right) Y^*_t\right)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{D.37}$$

$$C^*_t = \left[(1-\alpha)^{\frac{1}{\theta}} \left(\left(1-h_h(y_t)\right) Y_t\right)^{\frac{\theta-1}{\theta}} + \alpha^{\frac{1}{\theta}} \left(h_f(y_t) Y^*_t\right)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{D.38}$$

Let us now focus on further variables of interest for asset pricing: equity prices, and wealth.

$$Q_t = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u}{\xi_t} p_u Y_u du\right] \tag{D.39}$$

$$Q^*_t = \mathbb{E}_t \left[\int_t^\infty \frac{\xi^*_u}{\xi^*_t} p^*_u Y^*_u du\right] \tag{D.40}$$

$$W_t = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u}{\xi_t} P_u C_u du\right] \tag{D.41}$$

$$W^*_t = \mathbb{E}_t \left[\int_t^\infty \frac{\xi^*_u}{\xi^*_t} P^*_u C^*_u du\right] \tag{D.42}$$

ξ_t, ξ^*_t are the stochastic discount factors for the home and the foreign agent

$$\xi_t \equiv e^{-\rho t} P_t^{-1} C_t^{-\gamma} \tag{D.43}$$

$$\xi^*_t \equiv e^{-\rho t} P_t^{*-1} C_t^{*-\gamma} \tag{D.44}$$

In this complete-market world, they are related by the following relation

$$\xi_t = \phi \xi_t^* = \left(\frac{1-\lambda}{\lambda} \right) \xi_t^* \tag{D.45}$$

which is nothing but equation (D.20) above, i.e. the Backus-Smith condition.

From here, I then obtain ODEs for the following functions (in fact I obtain it for $J_t = \frac{P_t^\psi C_t}{W_t}$ to match the decentralized solution)

$$F_t^{-1} \equiv \frac{Q_t}{p_t Y_t} = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u}{\xi_t} \frac{p_u}{p_t} \frac{Y_u}{Y_t} du \right] \tag{D.46}$$

$$F_t^{*-1} \equiv \frac{Q_t^*}{p_t^* Y_t^*} = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u^*}{\xi_t^*} \frac{p_u^*}{p_t^*} \frac{Y_u^*}{Y_t^*} du \right] \tag{D.47}$$

$$J_t^{-1} \equiv \frac{W_t}{P_t C_t} = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u}{\xi_t} \frac{P_u}{P_t} \frac{C_u}{C_t} du \right] \tag{D.48}$$

$$J_t^{*-1} \equiv \frac{W_t^*}{P_t^* C_t^*} = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u^*}{\xi_t^*} \frac{P_u^*}{P_t^*} \frac{C_u^*}{C_t^*} du \right] \tag{D.49}$$

After deriving and solving those ODEs, the equilibrium obtained is the same as the one from the decentralized solution under CRRA preferences.

Solving for the equilibrium using the planner could be extended to recursive preferences following the approach in Dumas et al. (2000).

E. Extensions

E.1. Extension 1: global asset manager and the Global Financial Cycle (Sauzet, 2020a)

From the perspective of modeling the international financial system, an aspect that is increasingly being recognized as primordial is the role of global financial intermediaries. Those global intermediaries can be involved in the dealing of foreign currencies, in the spirit of Hau and Rey (2006) and Gabaix and Maggiori (2015), can play the role of bankers as in Maggiori (2017) and Jiang et al. (2020), or can play the role of global asset managers, like below. The main intuition is that because of their different preferences and limited risk-bearing capacity, the capitalization of those financial intermediaries is a prime determinant of asset prices, interest rates, exchange rates, and other economic outcomes worldwide. The presence of such global intermediaries is not only relevant from the perspective of realism, but could introduce a mechanism through which to capture additional aspects of the Global Financial Cycle of Rey (2013) and Miranda-Agrippino and Rey (2020), pertaining to the leverage and role of intermediaries. By way of an example, I briefly present one, the addition of a global asset manager, that I am exploring in ongoing work Sauzet (2020a). Figure F.2 summarizes the set-up.

The global asset manager constitutes a third type of investor, whose preferences, albeit still recursive and over the two goods, have the following specificities: (i) because she is a global citizen, the global asset manager has no particular bias towards any of the goods, and (ii) she is significantly more risk-tolerant than the consumer-investor of each country. The last point is in the spirit of the intermediary asset pricing literature, which typically models bankers as agents with lower risk aversion. Even though the current version of this work does not feature them, the limited risk-bearing capacity of the global asset manager, in the form for instance of portfolio constraints, will be an important addition.

The equilibrium can be represented as a function of three state variables, $X_t \equiv (x_t, y_t, u_t)'$. x_t is the wealth share of the domestic investor and is defined as before with the caveat that now, $W_t + W_t^*$ does not sum up to total world wealth, which is $W_t + W_t^* + W_t^{glam}$ and includes the wealth of the global asset manager W_t^{glam}. y_t still captures the relative supply of the goods. u_t, the new state variable, captures the share of world wealth held by the global asset manager.[89] In summary:

$$x_t \equiv \frac{W_t}{W_t + W_t^*} \quad ; \quad y_t \equiv \frac{Y_t}{Y_t + Y_t^*} \quad ; \quad u_t \equiv \frac{W_t^{glam}}{W_t + W_t^* + W_t^{glam}} \tag{E.1}$$

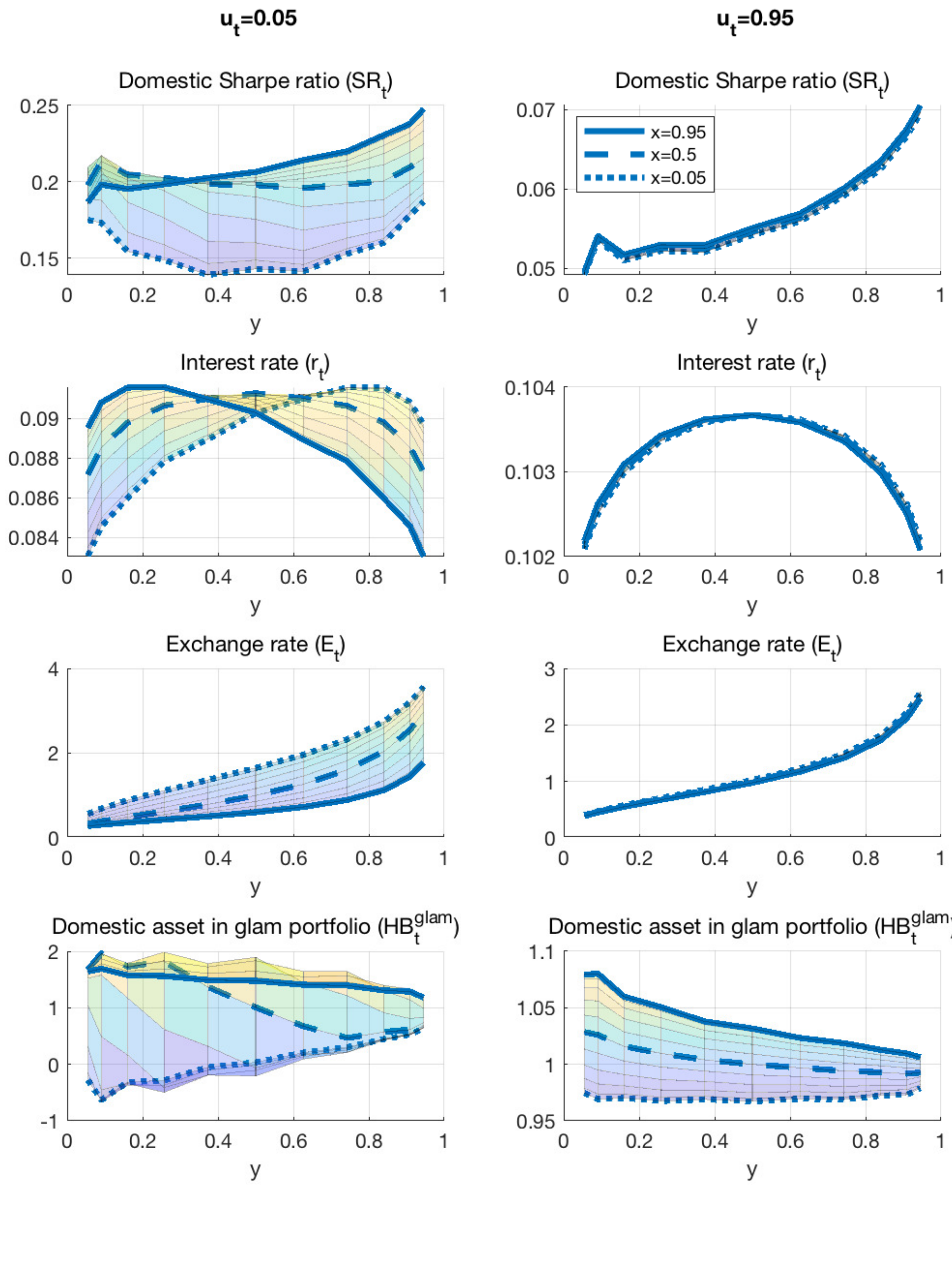
Equations are presented in Sauzet (2020a), and Figure E.1 shows the results. The preference heterogeneity of the global asset manager, coupled with that of the investor of each

[89] The share of the domestic and foreign investor in world wealth are now obtained as $x_t(1-u_t)$ and $x_t u_t$.

country, is able to generate rich patterns in global asset prices, interest rates, goods prices, and portfolios, even without portfolio constraints. For instance, the Sharpe ratio on the domestic asset is much larger when the global asset manager is poorly capitalized (u_t small), reflecting the higher compensation for risk required by the domestic and foreign consumer-investors to hold the domestic equity asset. This is also true for foreign equity, and points to the fact that a poorly capitalized global asset manager, a proxy more generally for the global financial system, leads to increased risk premia throughout the world, in a pattern reminiscent of a Global Financial Cycle. This mechanism could complement the one stemming from the role of the domestic country as world banker discussed in the main application in Section 4.3, by introducing financial intermediaries in the picture. When this happens, the risk premia on equity assets are also more dependent on the repartition of wealth across the remaining investors, captured by x_t, consistent with a crisis situation in which the identity of the average holder of an asset matters more and assets rapidly changing hands are accompanied by large swings in returns. The capitalization of the global asset manager also matters for interest rate, which tends to decrease as u_t gets small, reflecting a lower average risk tolerance in the economy, which corresponds with a higher demand for the safe asset (the international riskless bond). Goods prices are also affected, with the exchange rate depending significantly more on the allocation of wealth across consumer-investor. Note also the impact on portfolios: not only is the portfolio of the global asset manager getting further from the market portfolio as u_t decreases, but it is also increasingly affected by the allocation of wealth among the remaining consumer-investors. This reflects the fact that because she is not biased towards any particular asset, the global asset manager is here to pick up the opposite side of the trades for the other two investors, and this leads to wild changes in her portfolios especially as she gets less well-capitalized.

This brief illustration shows the promise of introducing global financial intermediaries in the framework of this paper, and highlights how it can complement the mechanisms discussed previously in the main application.

Figure E.1: Equilibrium in the presence of a global asset manager



Notes: Calibration: $\gamma^{glam} = 2 < \gamma = \gamma^* = 8, \psi = 0.2, \alpha = 0.85, \rho = 1\%$. x_t is the wealth share of the domestic investor as a fraction of $W_t + W_t^*$. y_t is the relative supply of the domestic good, which captures fundamentals. u_t is the share of world wealth held by the global asset manager.

E.2. Extension 2: towards a solution to the reserve currency paradox (Sauzet, 2020c)

In addition to global financial intermediaries, further extensions of the framework could help make way towards resolving the so-called "reserve currency paradox" emphasized by Maggiori (2017) and to which the reader is referred for details. The paradox appears as follows in the framework of my paper. Consider again that the domestic country represents the United States, the risk-tolerant country at the center of the international financial system. As we have seen, and consistent with Gourinchas et al. (2017): in normal times, the country enjoys an exorbitant privilege by earnings higher returns on average due to its riskier position, but in crisis times, it bears the exorbitant duty of insuring the rest of the world through a wealth transfer. In turn, because of the home bias in consumption, this wealth transfer towards the rest of the world tends to increase the price of foreign goods, which pushes up the price of the foreign basket and lead the domestic currency, the US dollar in this case, to depreciate. The reserve currency paradox resides in the fact that this is clearly counterfactual: empirically, the US dollar tends to *ap*preciate in crisis, which is one of the main reasons why it is the world's major reserve currency in the first place. As discussed in Maggiori (2017), this paradox does not depend on the specifics of the underlying model – for instance, the framework in this paper is quite different from his. Instead, it is deeply rooted in the presence of the home bias in consumption, an aspect that goes back all the way to the classical "transfer problem" of Keynes and Ohlin discussed previously.

Maggiori (2017) presents a potential resolution based on trade costs depending negatively on the capitalization of financial intermediaries. Another part of the story, that I plan to implement in the current framework, relies on the importance of trade in bonds. Specifically, times of crisis are periods in which the demand for safe assets usually skyrockets ("risk-off" episodes). Because the United States is the main provider of safe asset worldwide, this sudden increase in the demand for US Treasuries goes hand-in-hand with a strong upward pressure on the currency in which they are denominated. This, in my view, is one of the main ultimate drivers of US dollar appreciation in times of crisis. To introduce such channels in the framework developed in this paper, I plan to include the following elements in future extensions (Sauzet (2020c), ongoing). First, the demand for safe asset must be meaningfully time-varying, which I plan to generate from time-varying risk aversion in the form of heterogeneous investors with varying degrees of risk aversion *within countries*. A risk-off episode would therefore correspond to an event in which the risk-tolerant investor of a country is poorly capitalized. Second, the bond of the center country should be particularly attractive in difficult times[90], which could come from an ad-hoc feature or potentially by assuming that the size of the center country is larger so that its bond ensures against a larger share of world shocks, in the spirit of Hassan (2013).[91] Third, for this "trade in

[90] A related and subtle point is to disentangle the extent to which the upward pressure on US Treasuries is itself driven by the safety of the US dollar in times of crisis.

[91] To do this, reformulating the output share y_t by adapting the share process of Menzly et al. (2004); Santos and Veronesi (2006) as mentioned previously could be particularly useful.

assets" channel to matter enough for exchange rates so as to reverse the reserve currency paradox driven by the trade in goods, the introduction of global financial intermediaries will be important quantitatively. They could take the form of global asset managers as presented above, intermediating trade in assets, or of global foreign currency dealers in the spirit of Hau and Rey (2006) and Gabaix and Maggiori (2015). Their role would be to ensure that, like in practice, the increased demand for bonds is met with limited capacity, which ultimately leads to an upward pressure on the price of the US currency. Finally, the introduction of portfolio constraints, for both global intermediaries and for the different investors within each country, as well as other sources of market incompleteness, will also prove important for the mechanism to have bite quantitatively.

E.3. Extension 3: projection methods via neural networks (Sauzet, 2020b)

The extensions above make clear that the number of state variables is likely to rapidly increase with additions to the framework. Projection methods are conceptually well-suited to contexts with multiple state variables, and are typically better able to handle a larger number of them than other approaches like finite-difference methods, which become rapidly computationally too costly.[92] As a result, they are well-adapted to the environment in this paper. To be sure however, computationally, traditional projection methods also are very much subject to the curse of dimensionality, and scaling the number of state variables further up will prove limited using standard Chebyshev polynomials. For instance, even the addition of a third state variable, like in the global asset manager extension above, renders the resolution significantly slower, and increasing the order of approximation much beyond $N = 10$ proves difficult. More refined ways to construct the Chebyshev polynomials and corresponding grids, such as complete polynomials or Smolyak's algorithm, could help. Ultimately however, they are also limited and methods able to handle higher-dimensional cases will be required.

One such method consists in naturally extending the concept of projection approaches, but to replace the Chebyshev polynomials in the approximation by neural networks. In ongoing work (Sauzet, 2020b), I am developing these "projection methods via neural networks" to be applied to continuous-time problems like the one in this paper. The use not only of neural networks, but of the whole eco-system of related packages, proves of tremendous importance. First, those packages and environments, like Tensor Flow on which my implementation is based, are specifically designed for very high-dimensional contexts such as computer vision or other artificial-intelligence-type problems. As such, they are able to handle billions of observations and multiple millions of parameters. Even in the framework of this paper, this would allow me to focus on a much finer grid than do Chebyshev polynomials. Second, provided that one is judicious in the choice of the specification of the neural networks (typically in the choice of activation functions), they are naturally able to handle very non-linear functions. This aspect will prove particularly important when introducing portfolio constraints, which typically lead to sharp non-linearities, and are not necessarily handled well by Chebyshev polynomials especially of low order. Third, fitting neural networks conceptually in a projection framework is also particularly useful. Contrary to other methods based on neural networks that are more akin to value function iteration, e.g. Duarte (2019), a method expressed in a projection approach framework is able to naturally handle even cases for which value function iteration is difficult to adapt. For instance, economies with multiple agents and incomplete markets, for which there are several value functions as well as other unknown functions, would be difficult to cast in a value function iteration framework, but pose no particular problem for projection methods via neural networks.

Overall, the method has promise. For instance, I solve a "Ten Trees" equivalent to

[92] The method currently developed in Hansen et al. (2018) could potentially help from that perspective.

Cochrane et al. (2008)'s "Two Trees" without particular difficulty, a fit that would prove impossible for Chebyshev polynomials, and even less so for finite-different methods.[93]

[93]On this problem, Martin (2013) proposes an alternative method that proves promising even with five or six trees, and possibly more. The method also allows for jumps.

F. Additional Figures

F.1. Economic set-up

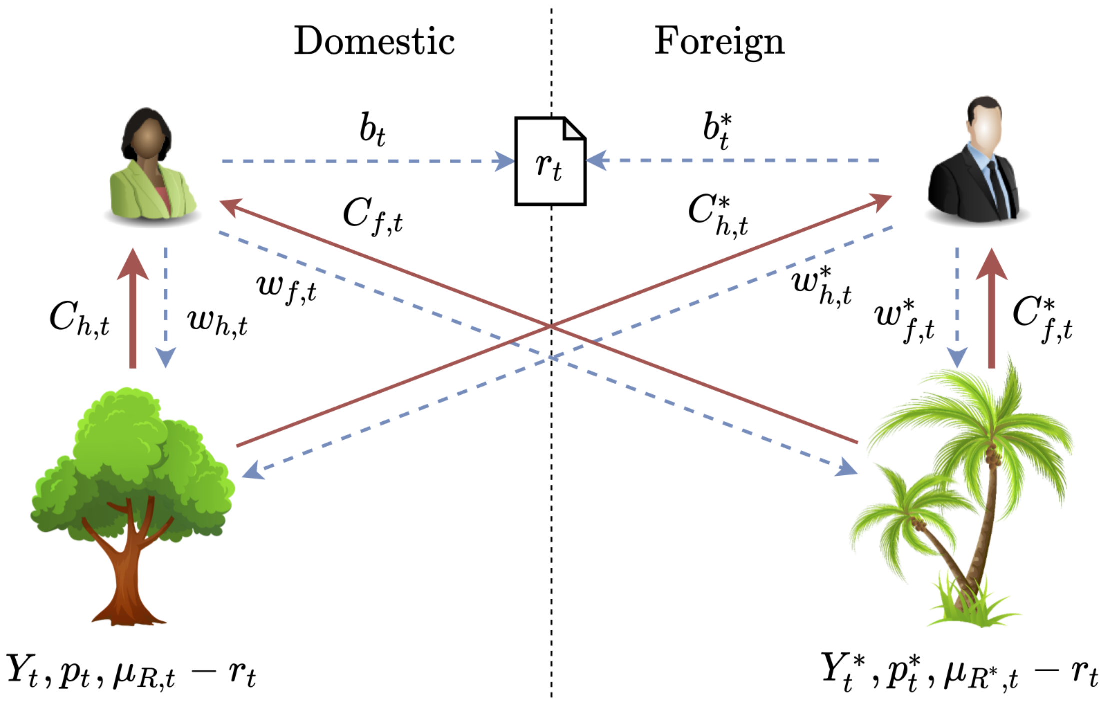


Figure F.1: Baseline international economy

Notes: Back to main text: Section 2.

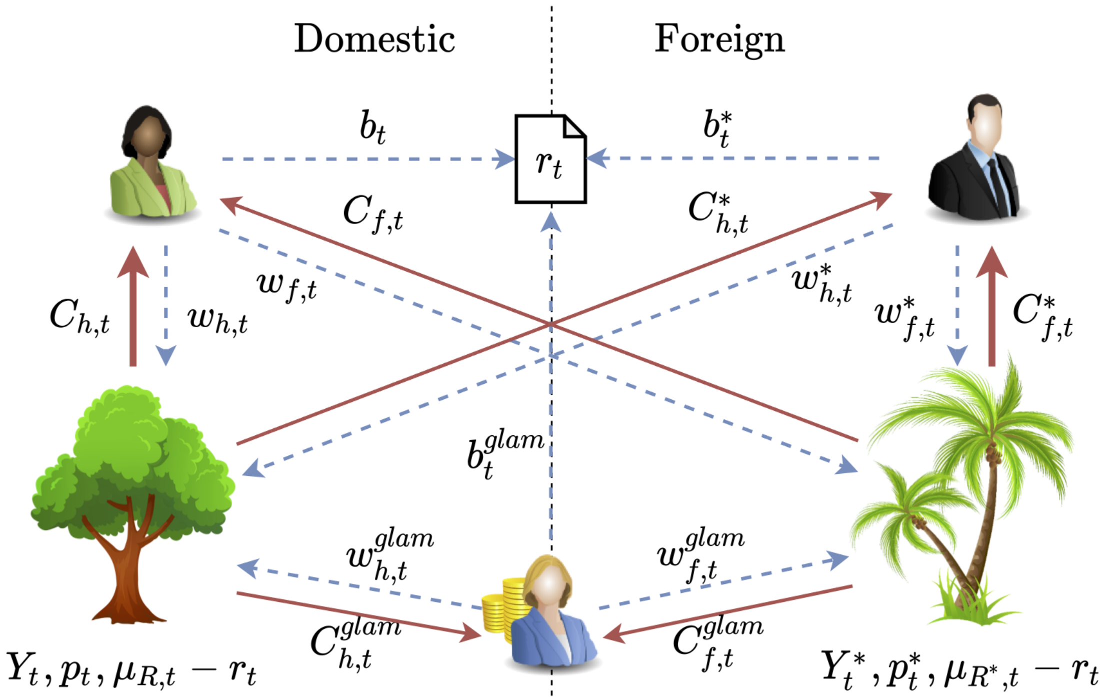


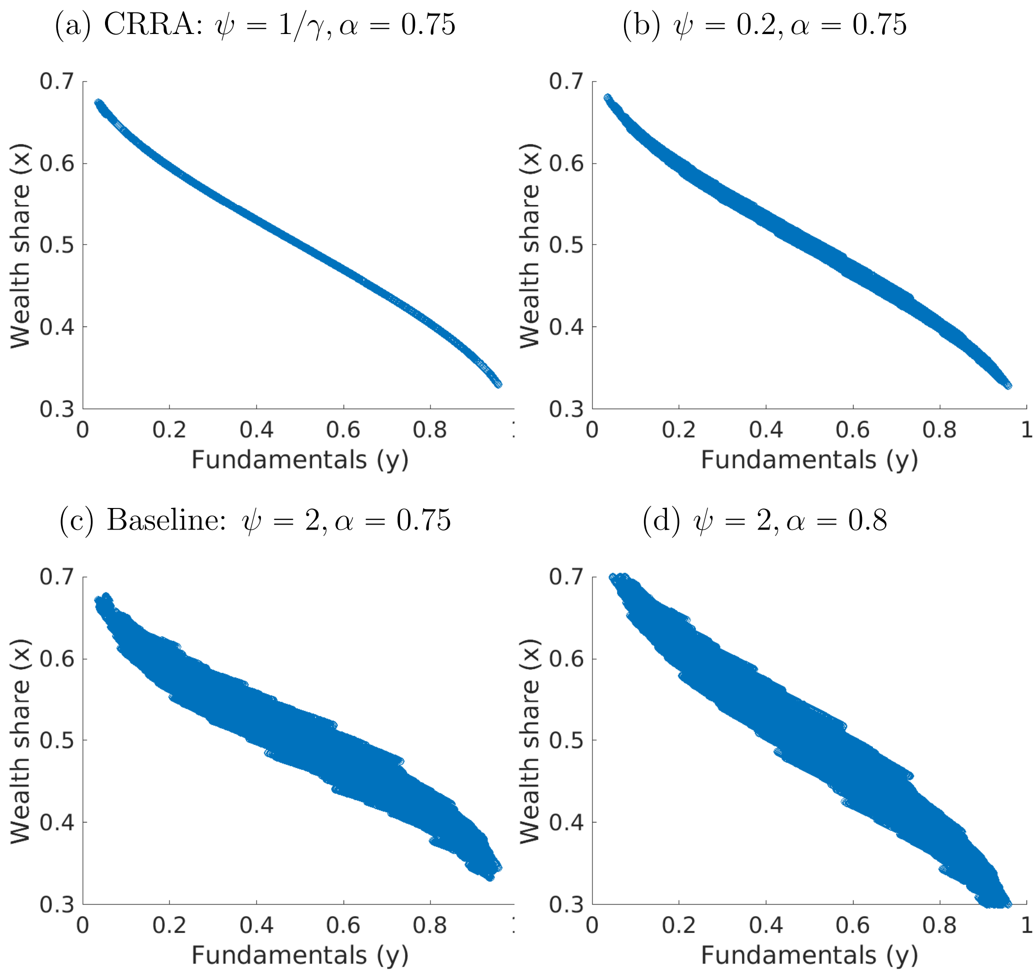
Figure F.2: International economy in the presence of a global asset manager

Notes: Back to main text: Section 4.5, back to Appendix: Section E.1.

F.2. Distributions

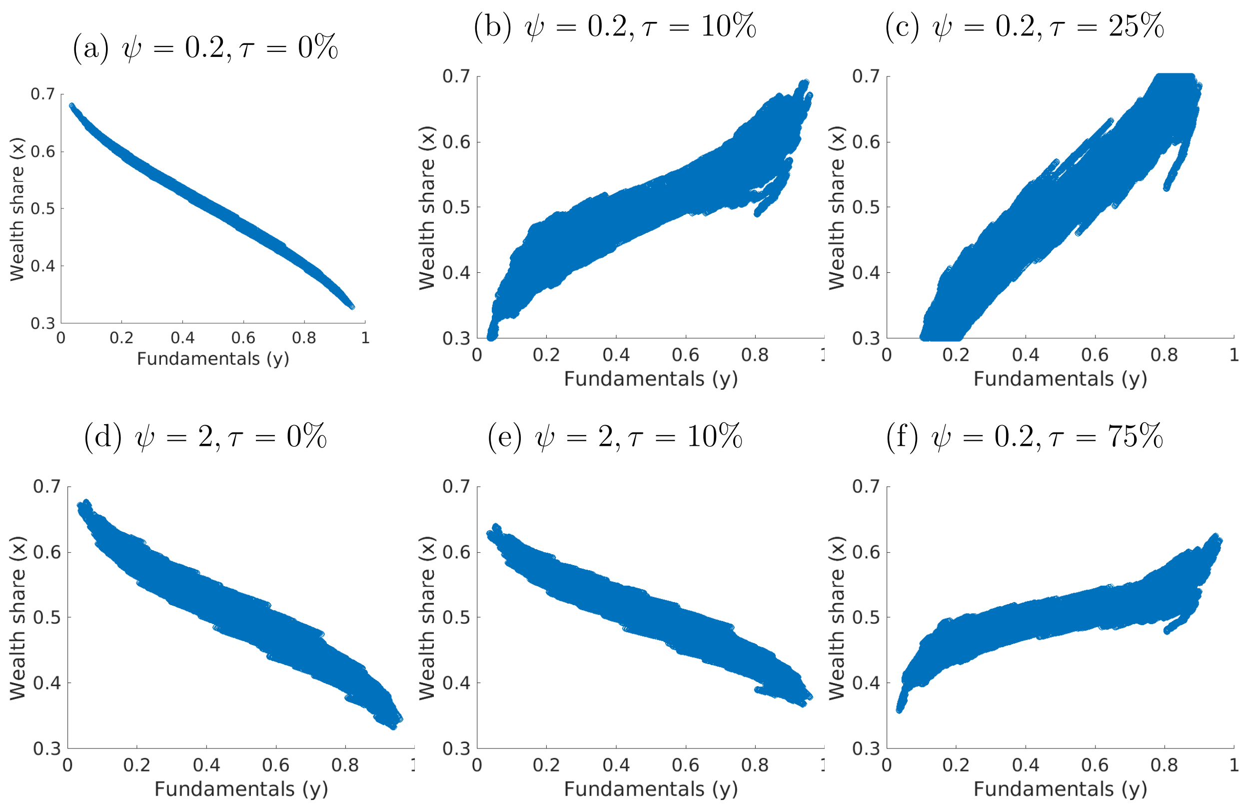
All distributions, unless otherwise specified are obtained from $nsim = 1,000$ paths of length $T = 250$ years, with $dt = 0.01$ (biweekly frequency), starting from $X_0 = (1/2, 1/2)$. The distributions are shown from the top, and for visibility each point visited during the simulation is shown with the same intensity.

Figure F.3: Distribution of the state variables in the baseline calibration



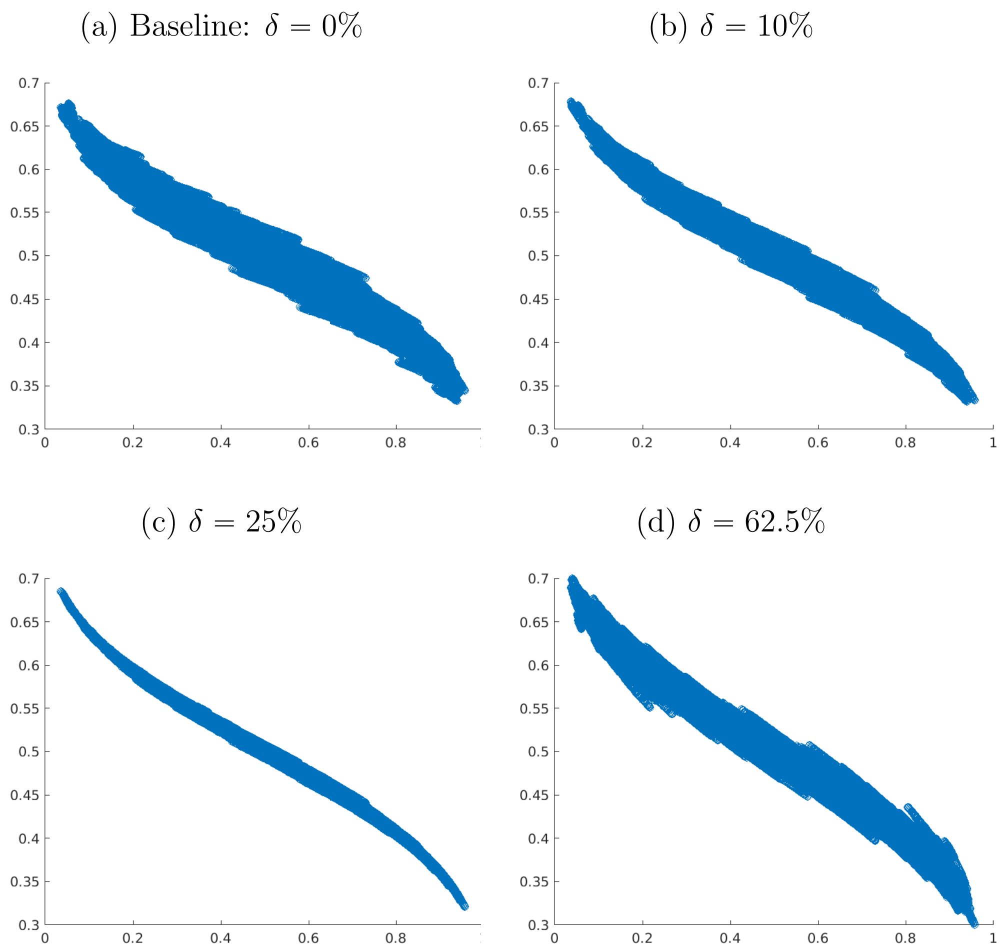
Notes: x_t, the wealth share, which captures the share of worldwide wealth held by the domestic investor, is shown on the vertical axis. y_t, the domestic output share, which captures fundamentals, is shown on horizontal axis. Distribution seen from the top, and obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

Figure F.4: Distribution of the state variables under imperfect financial integration



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1 (specifically $\psi = 2$), except for labor income μ. x_t, the wealth share, which captures the share of worldwide wealth held by the domestic investor, is shown on the vertical axis. y_t, the domestic output share, which captures fundamentals, is shown on horizontal axis. Distribution seen from the top, and obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

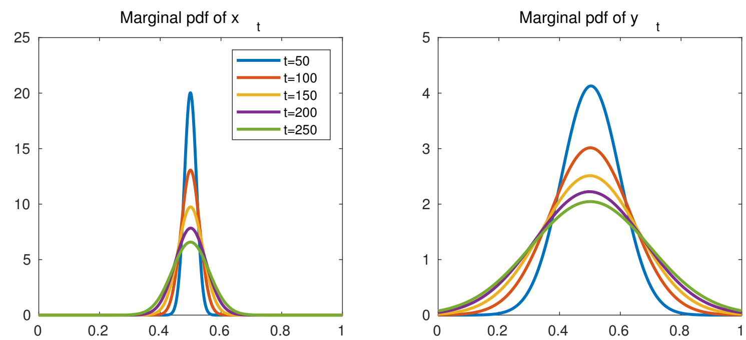
Figure F.5: Distribution of the state variables in the presence of labor income (δ)



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1 (specifically $\psi = 2$), except for labor income δ. x_t, the wealth share, which captures the share of worldwide wealth held by the domestic investor, is shown on the vertical axis. y_t, the domestic output share, which captures fundamentals, is shown on horizontal axis. Distribution seen from the top, and obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

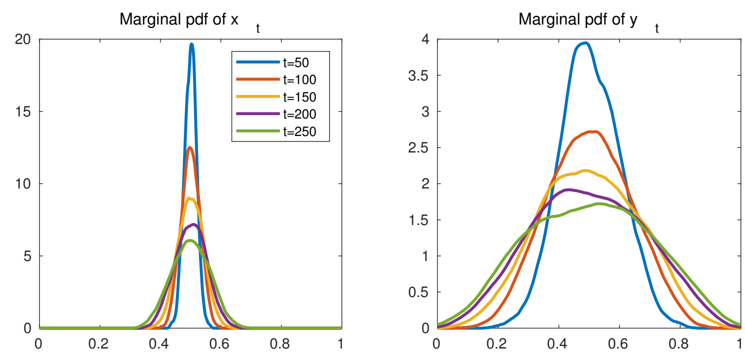
F.3. Evolution of the distribution of X_t over time

Figure F.6: Marginal distributions for x_t and y_t over time (Normal kernel, baseline calibration)



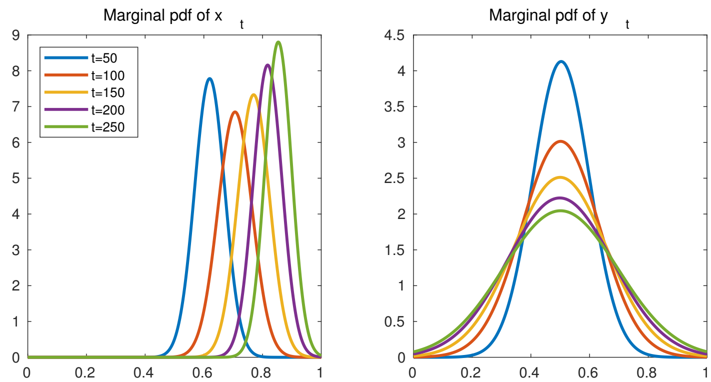
Notes: x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Distribution obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

Figure F.7: Marginal distributions for x_t and y_t over time (Epanechnikov kernel, baseline calibration)



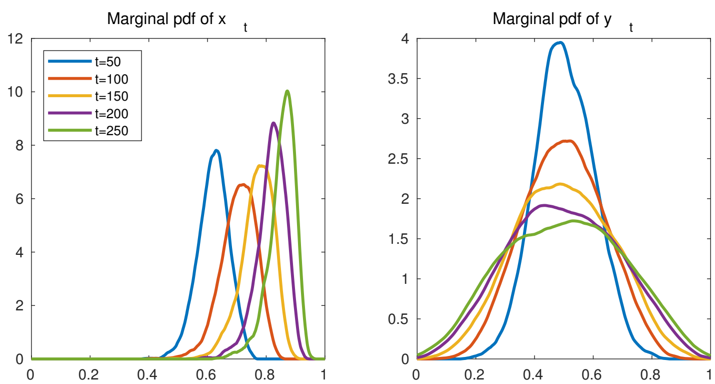
Notes: x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Distribution obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

Figure F.8: Marginal distributions for x_t and y_t over time (Normal kernel, $\gamma = 7.5 < \gamma^* = 15$)



Notes: x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Distribution obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

Figure F.9: Marginal distributions for x_t and y_t over time (Epanechnikov kernel, $\gamma = 7.5 < \gamma^* = 15$)



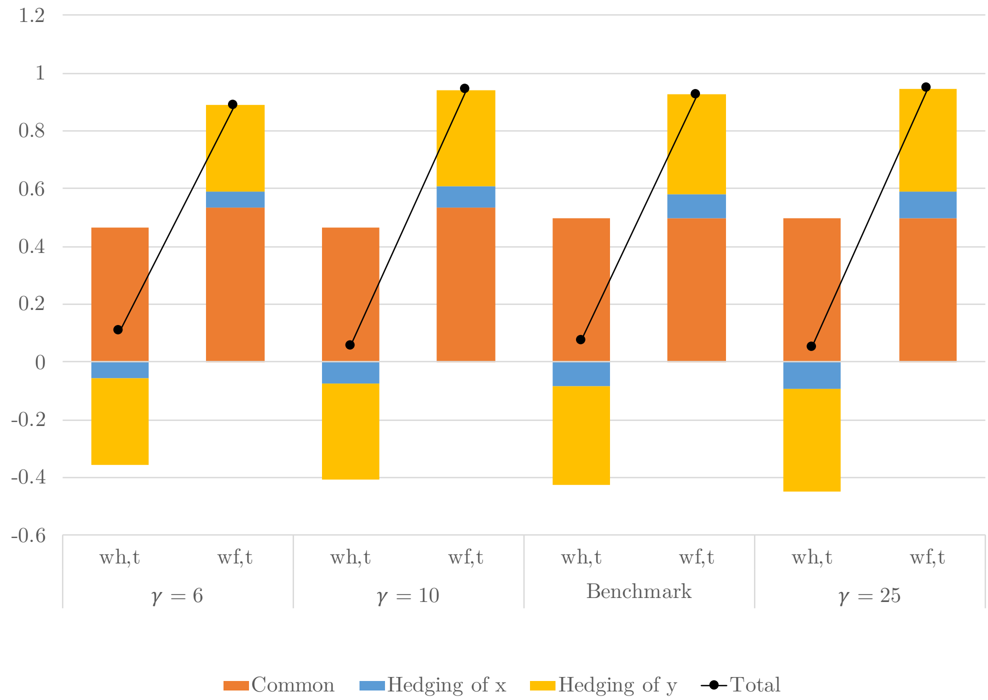
Notes: x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Distribution obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

F.4. Portfolios at the symmetric point

Figure F.10: Equity portfolio at $X_t = (1/2, 1/2)$ and home bias in consumption α

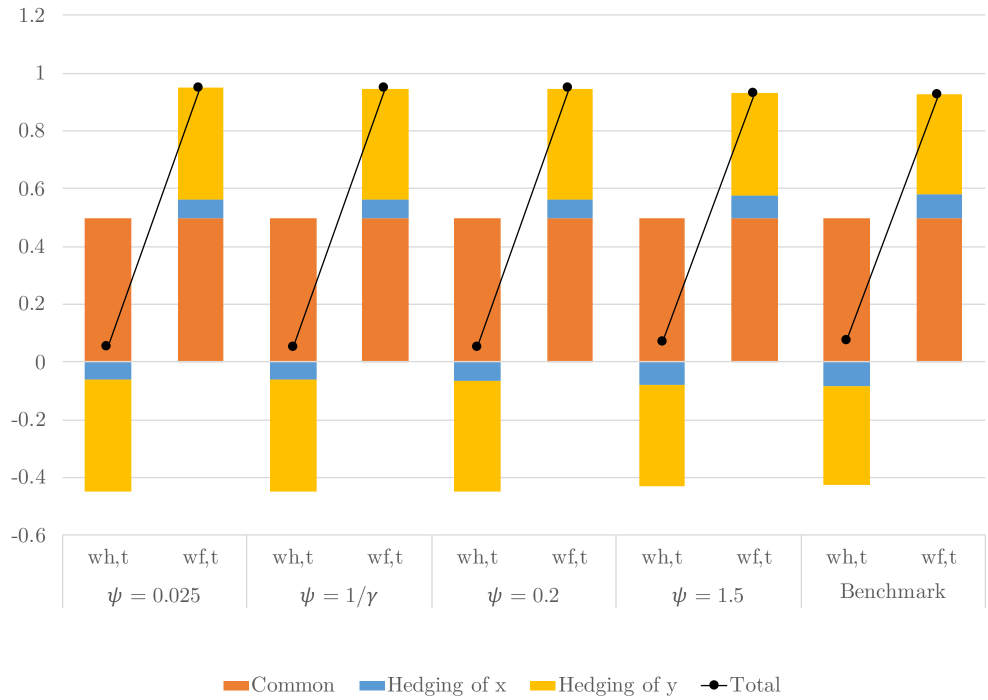
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except α. The figure shows portfolios when both the allocation of wealth (x_t) and the relative supply (y_t) are symmetric, $X_t = (1/2, 1/2)$.

Figure F.11: Equity portfolio at $X_t = (1/2, 1/2)$ and risk aversion γ



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except γ. The figure shows portfolios when both the allocation of wealth (x_t) and the relative supply (y_t) are symmetric, $X_t = (1/2, 1/2)$.

Figure F.12: Equity portfolio at $X_t = (1/2, 1/2)$ and elasticity of intertemp. substitution ψ

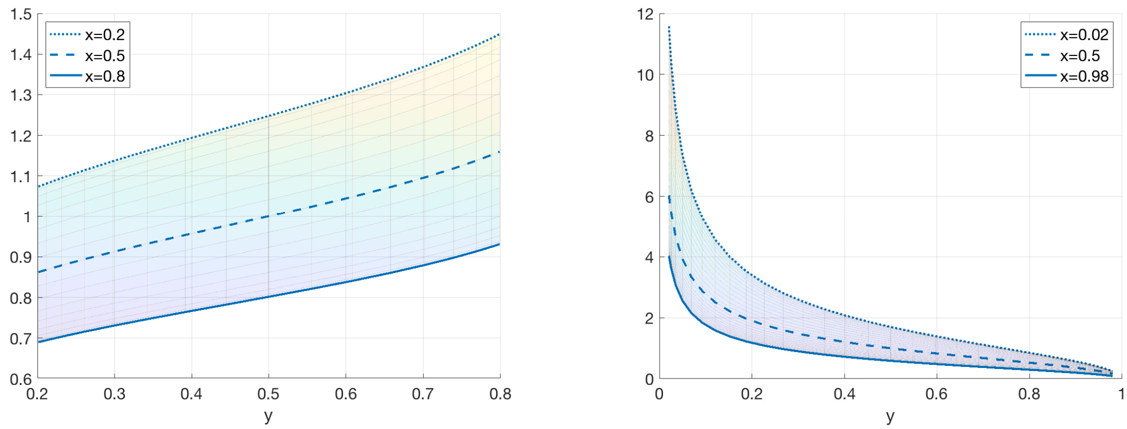


Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except ψ. The figure shows portfolios when both the allocation of wealth (x_t) and the relative supply (y_t) are symmetric, $X_t = (1/2, 1/2)$.

F.5. Representations as a function of both state variables

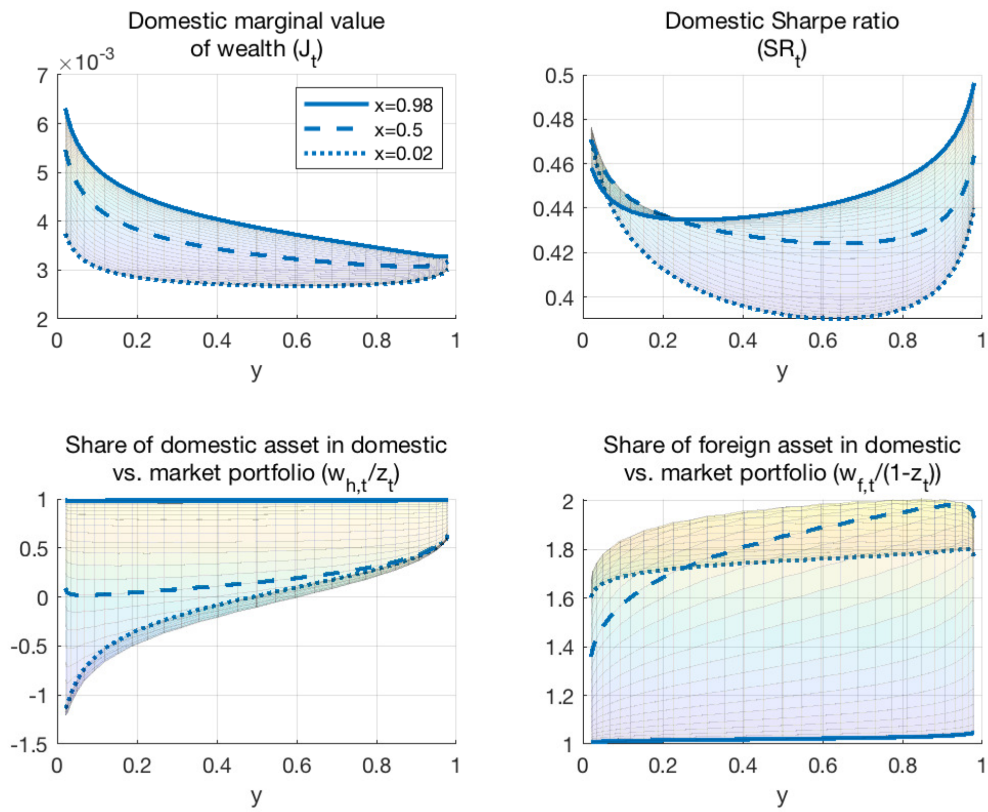
Figure F.13: Relative dividends: $p_t^* Y_t^* / (p_t Y_t)$

(a) $\theta = 0.9^* < 1$ (b) $\theta = 2 > 1$



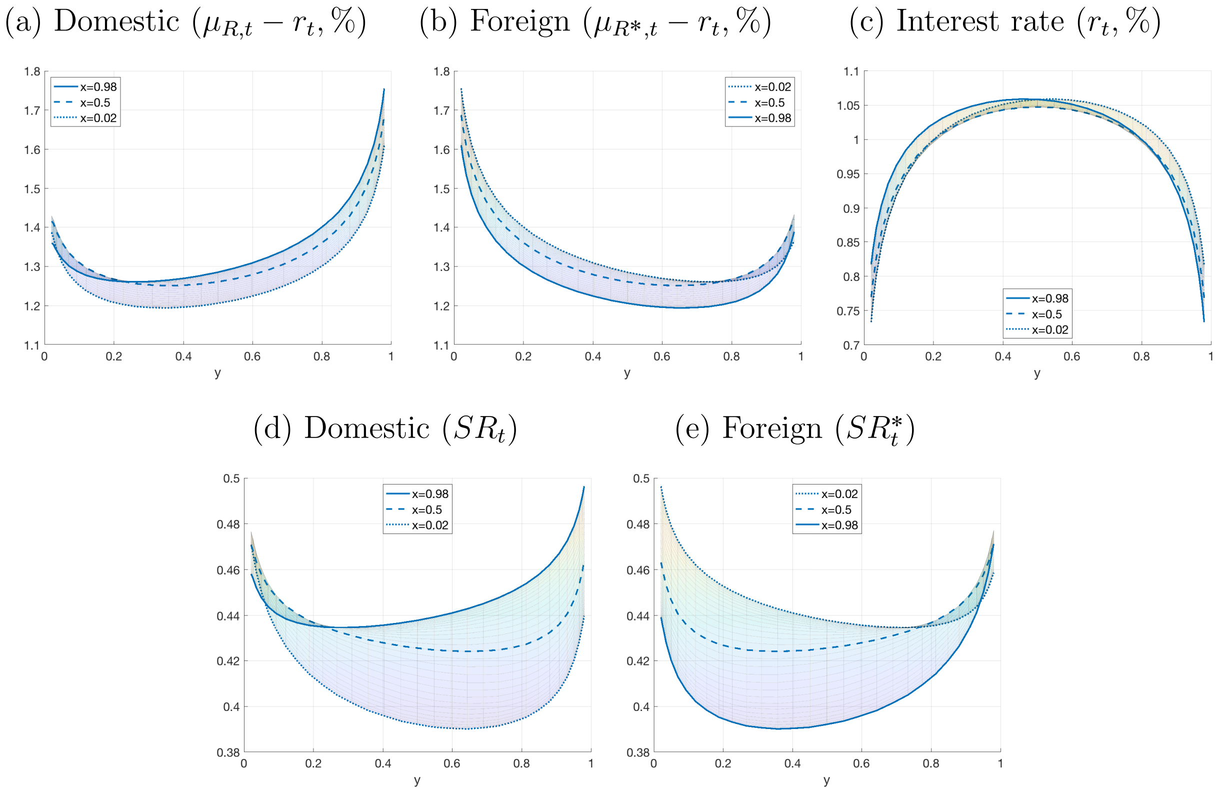
Notes: Based on the symmetric calibration of Assumption 1, except for the elasticity of substitution across goods, θ. * For Panel (a), $\gamma = 15, \psi = 1/\gamma, \alpha = 0.58$ (final calibration ongoing). x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.14: Direct impact of the wealth share



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding three-dimensional representations: Figure F.33.

Figure F.15: Expected risk premia, Sharpe ratios, and interest rate



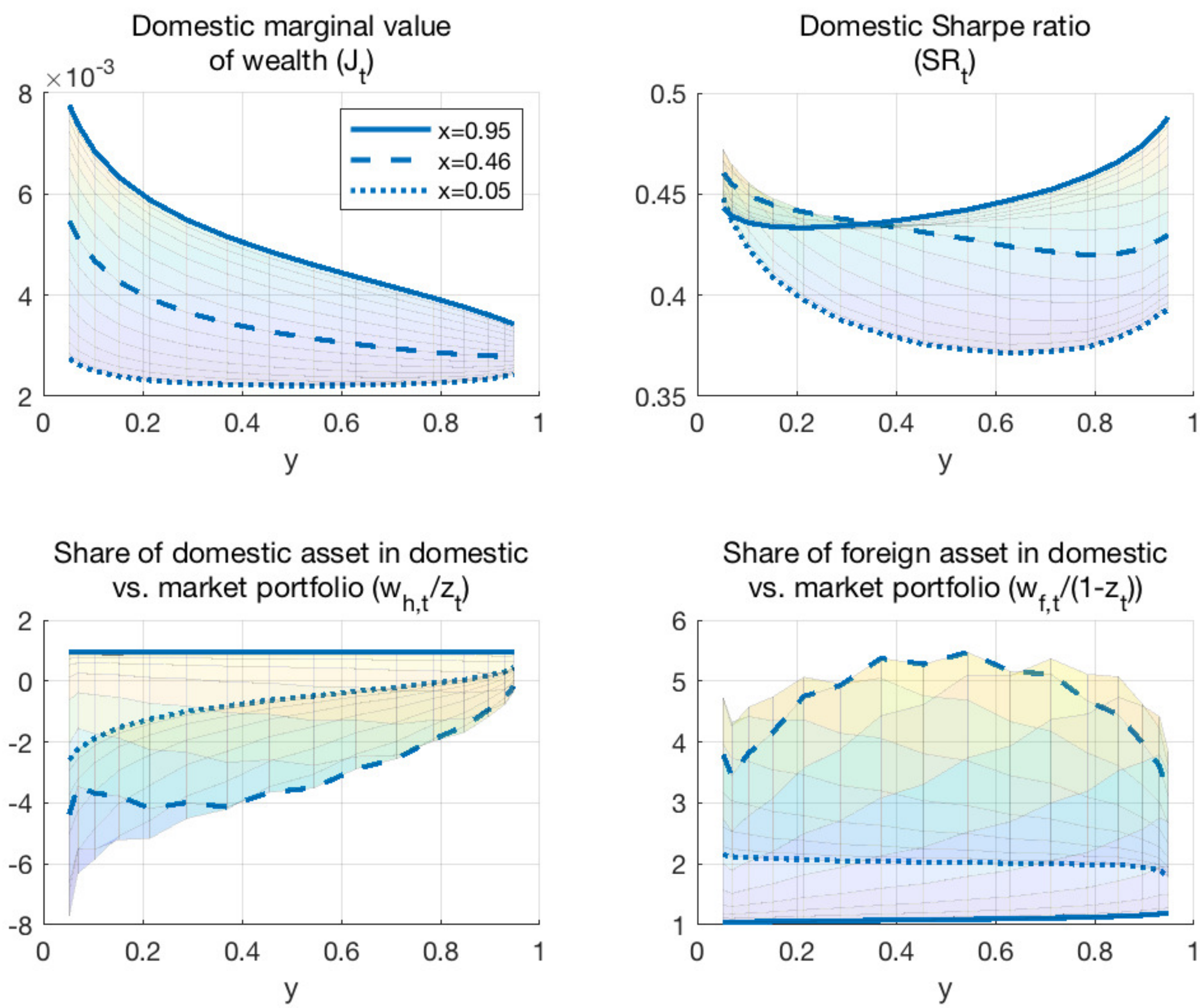
(a) Domestic ($\mu_{R,t} - r_t, \%$) (b) Foreign ($\mu_{R*,t} - r_t, \%$) (c) Interest rate ($r_t, \%$)

(d) Domestic (SR_t) (e) Foreign (SR_t^*)

Notes: Based on the symmetric calibration of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding representation when $x_t = 1/2$: Figure 4.

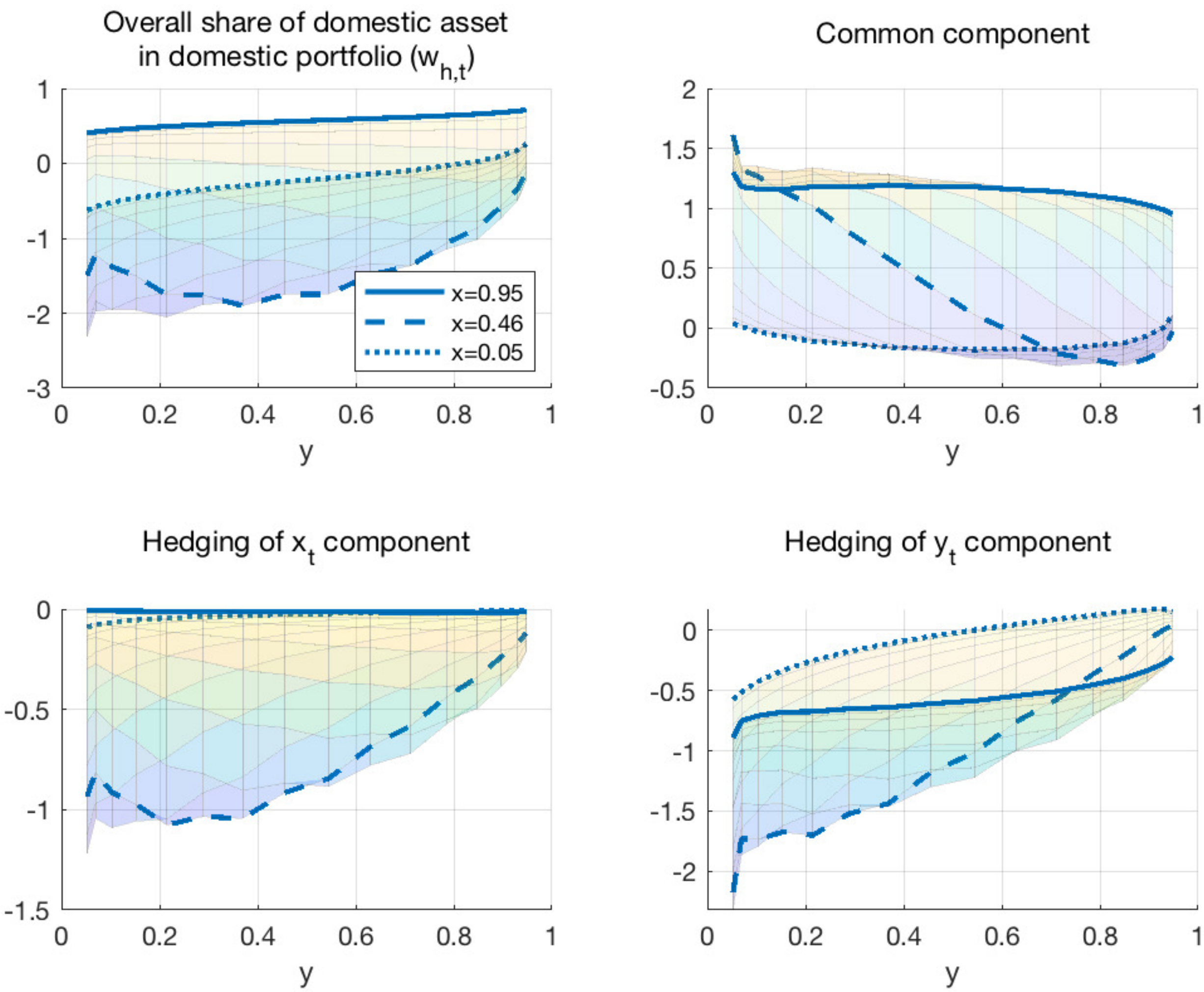
F.6. Effect of the home bias in consumption

Figure F.16: Direct impact of the wealth share for $\alpha = 0.85$



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except that $\alpha = 0.85$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

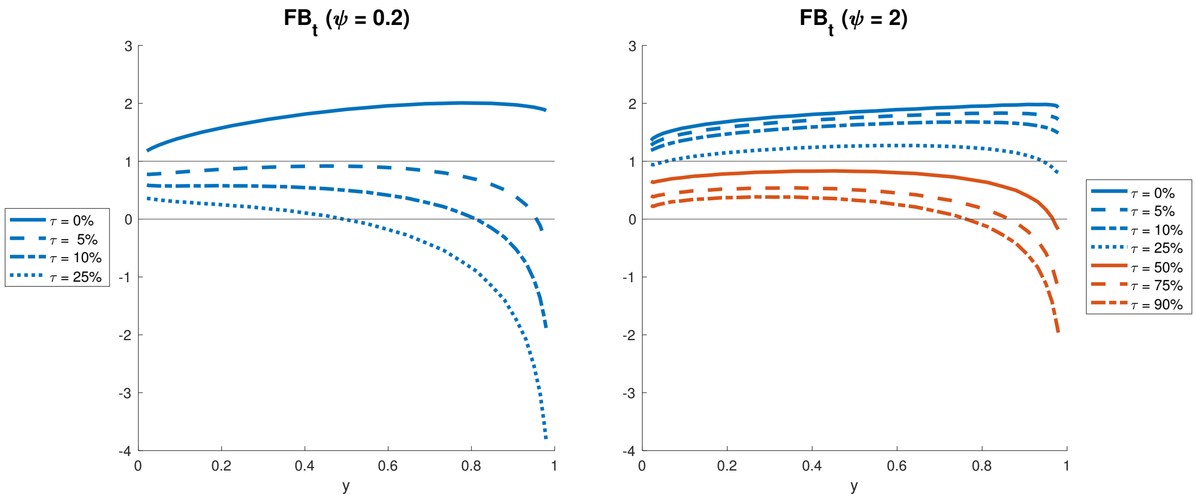
Figure F.17: Components of the domestic portfolio with $\alpha = 0.85$



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except that $\alpha = 0.85$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

F.7. Effect of imperfect financial integration

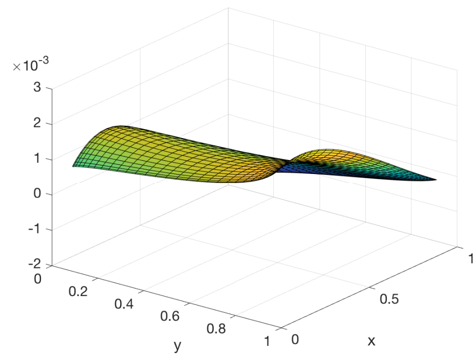
Figure F.18: Domestic equity portfolio vs. market portfolio



Notes: Based on the symmetric calibration of Assumption 1, except for ψ and τ. The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the domestic good, which captures fundamentals. Effect on the home bias measure HB_t: Figure 8.

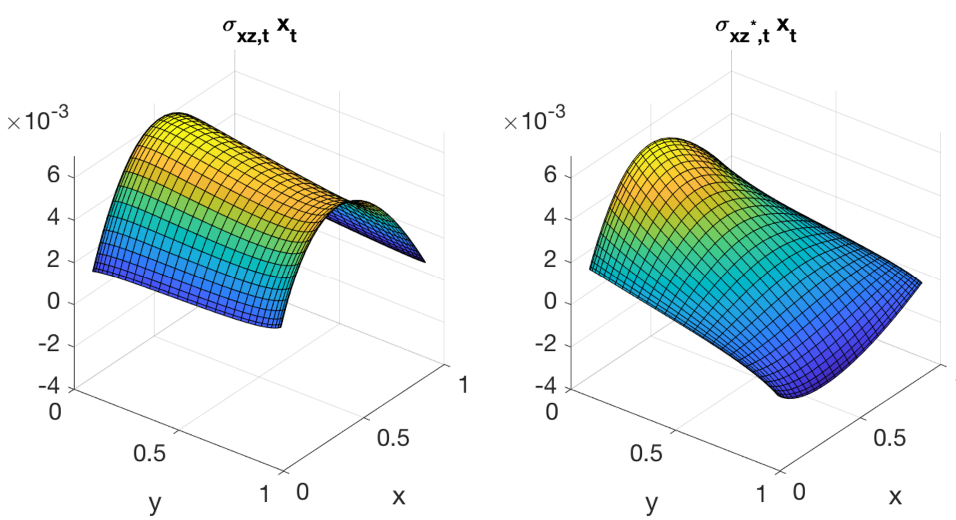
F.8. Application: The International Financial System

Figure F.19: Drift of the wealth share ($\mu_{x,t} x_t$)



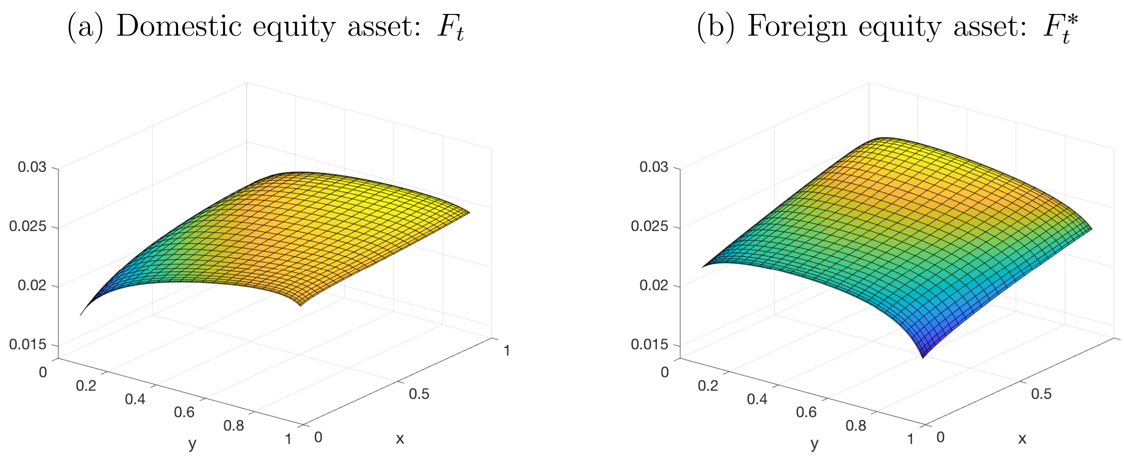
Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.20: Diffusion of the wealth share ($\sigma_{x,t} x_t$)



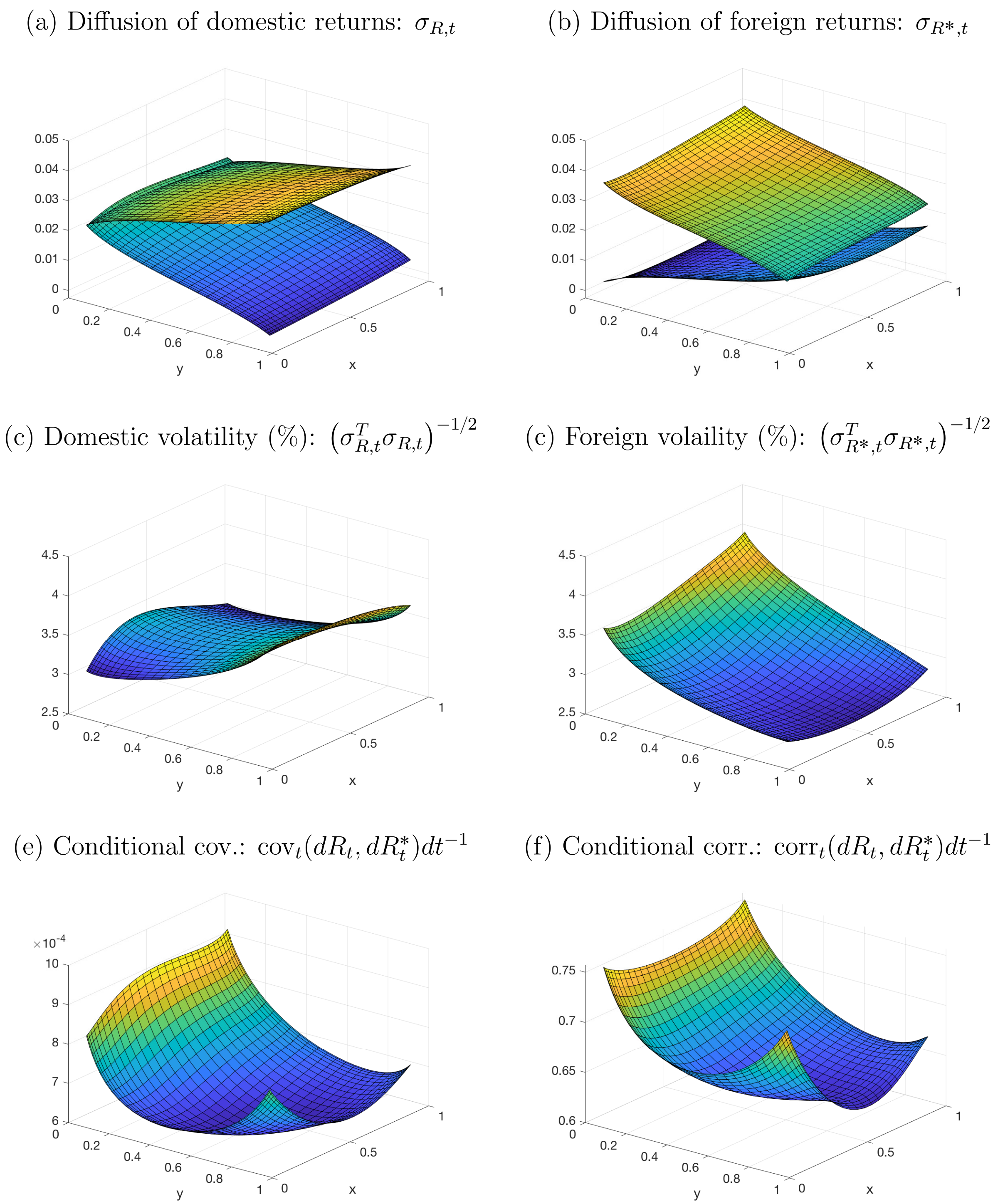
Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.21: Dividend yields in the application of Section 4



Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

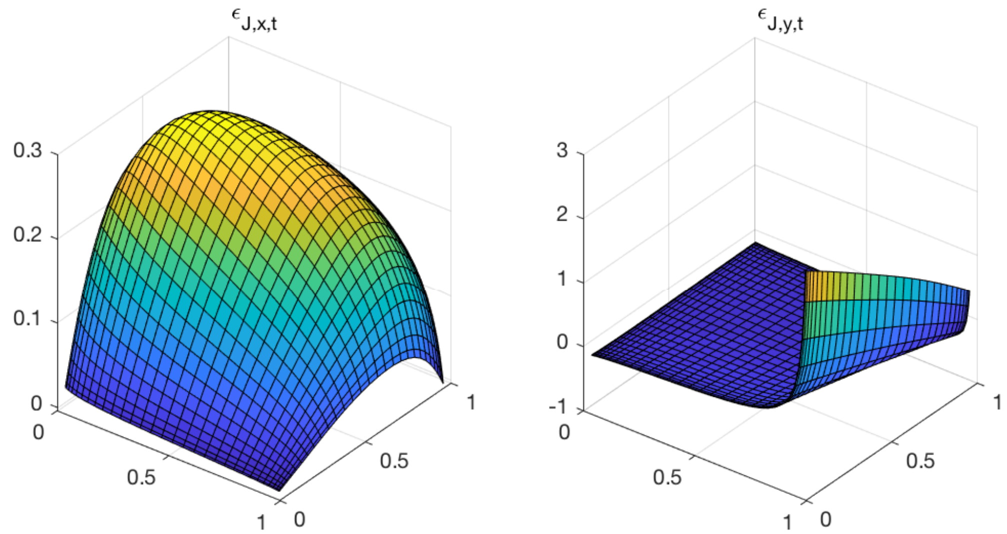
Figure F.22: Second moments of returns in the application of Section 4



Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

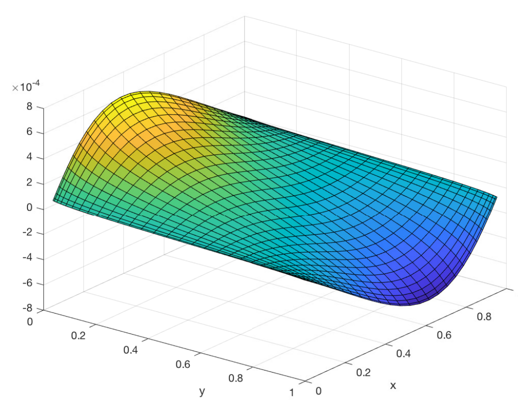
F.9. Other three-dimensional figures

Figure F.23: Conditional elasticities of the domestic marginal value of wealth (J_t)



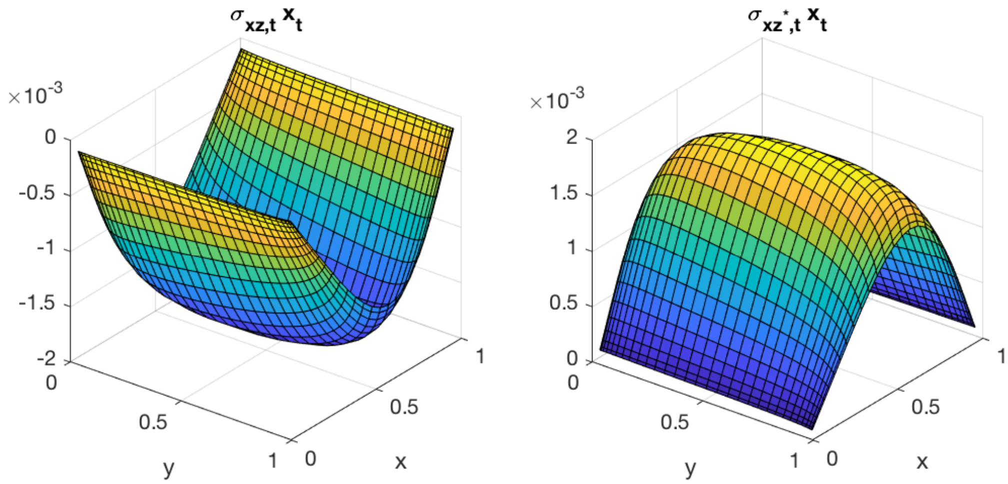
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.24: Drift of the wealth share ($\mu_{x,t}x_t$)



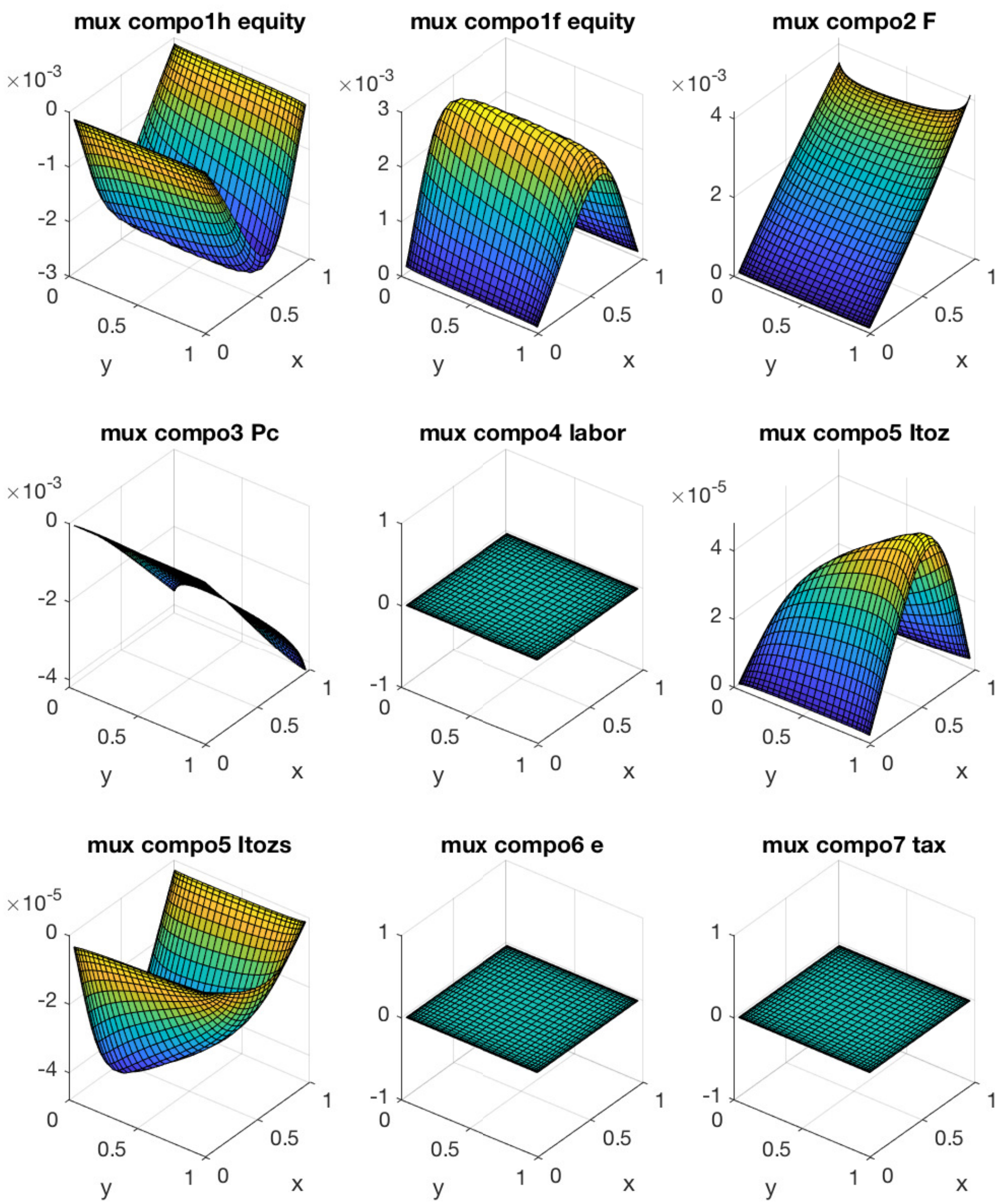
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.25: Diffusion of the wealth share ($\sigma_{x,t} x_t$)



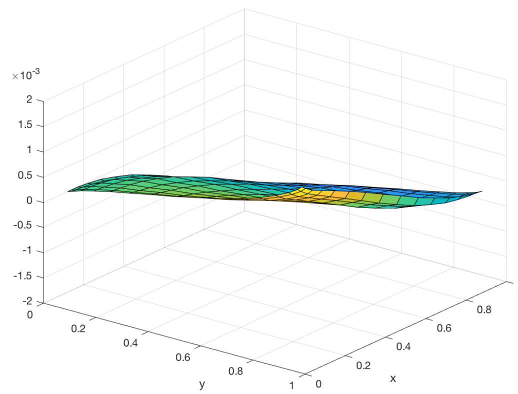
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding two-dimensional representation: Figure 1.

Figure F.26: Components of the drift of the wealth share ($\mu_{x,t} x_t$)



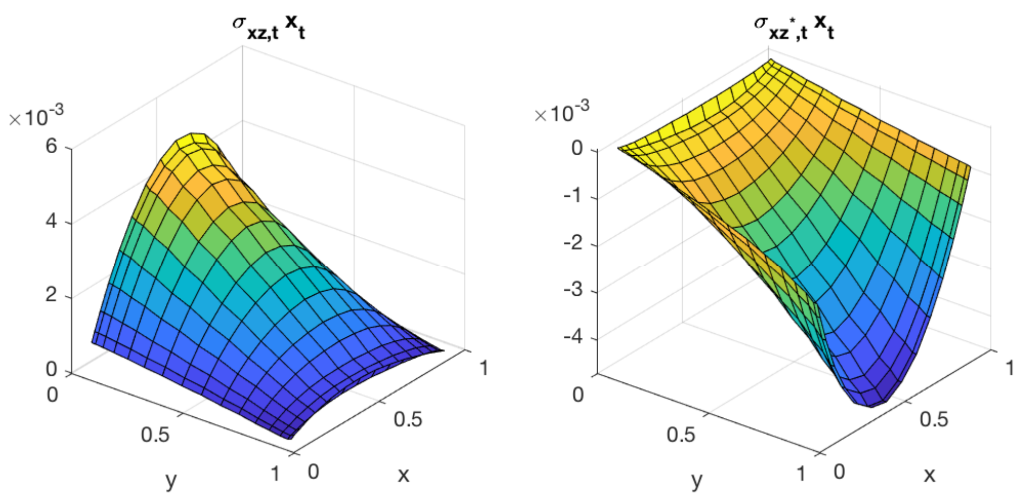
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.27: Drift of the wealth share ($\mu_{x,t}x_t$) under imperfect financial integration ($\psi = 0.2, \tau = 10\%$)



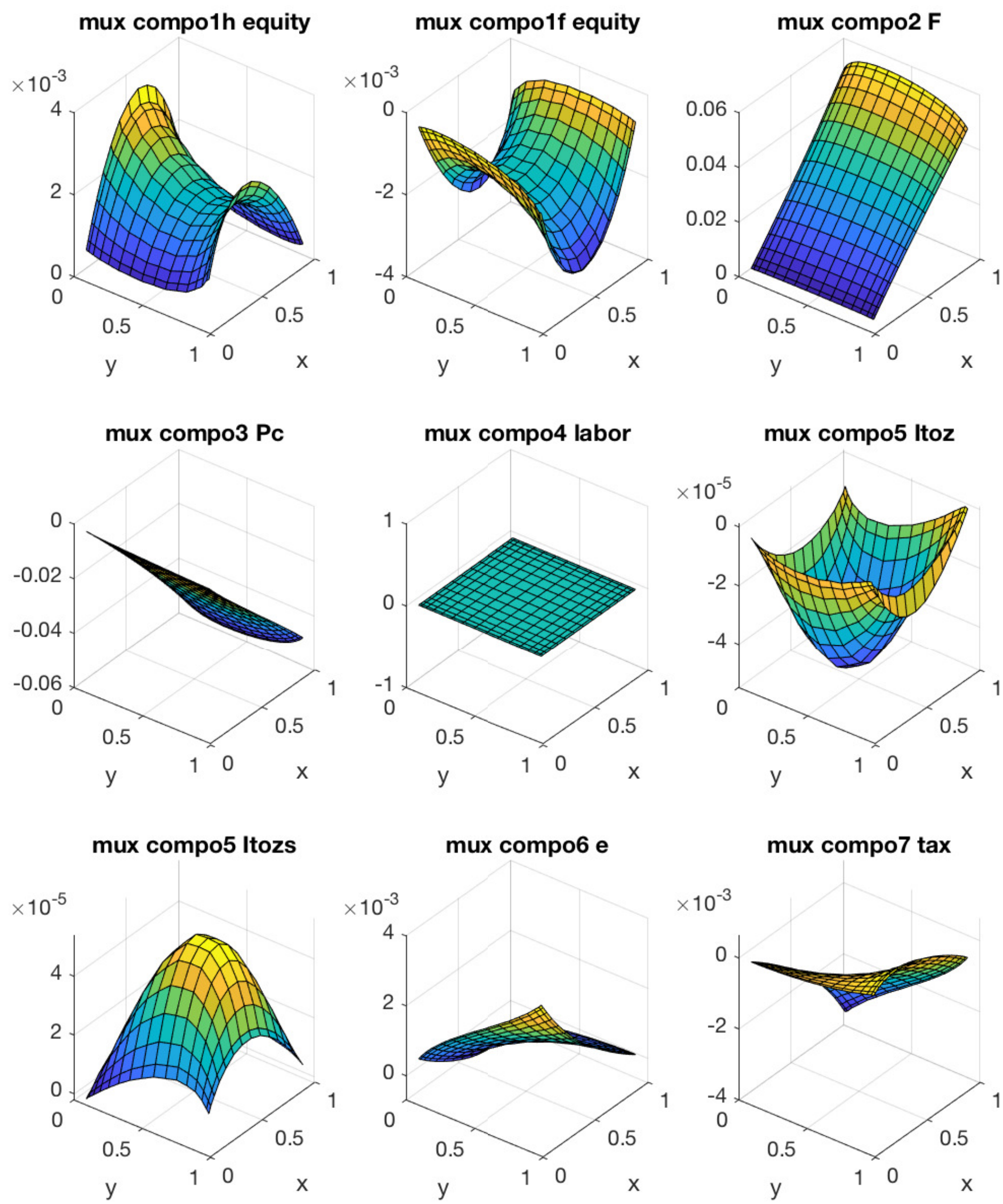
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.28: Diffusion of the wealth share ($\sigma_{x,t}x_t$) under imperfect financial integration ($\psi = 0.2, \tau = 10\%$)



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

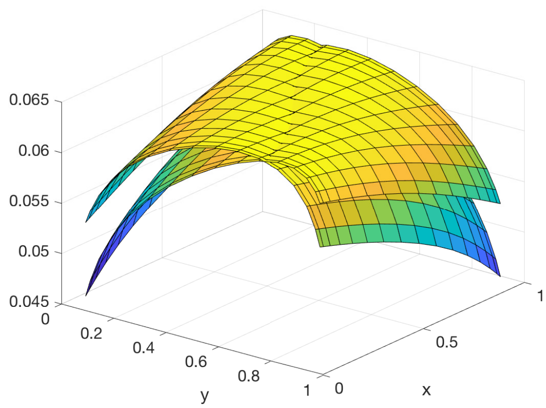
Figure F.29: Components of the drift of the wealth share ($\mu_{x,t}x_t$) under imperfect financial integration ($\psi = 0.2, \tau = 10\%$)



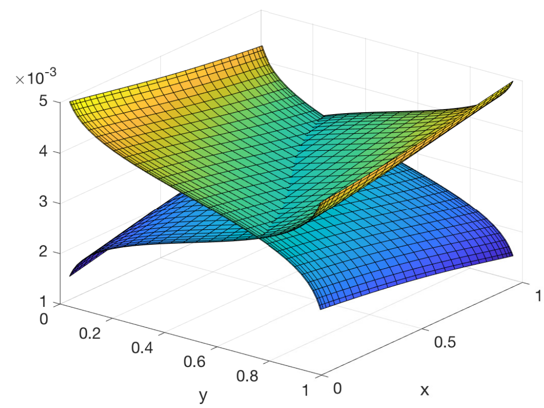
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.30: Dividend yields

(a) $\psi = 0.2$

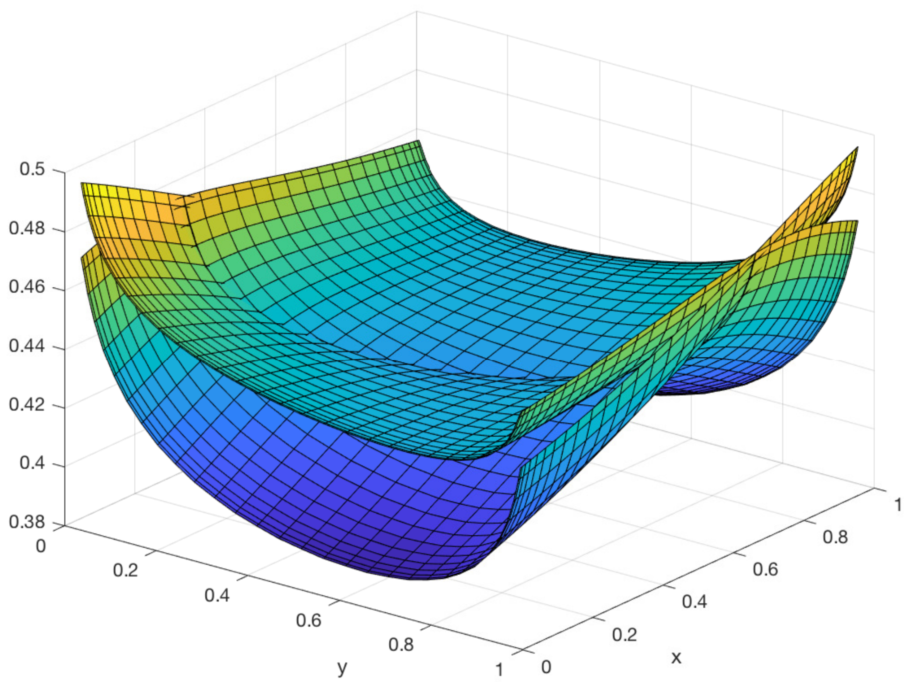


(b) $\psi = 2$



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.31: Sharpe ratios

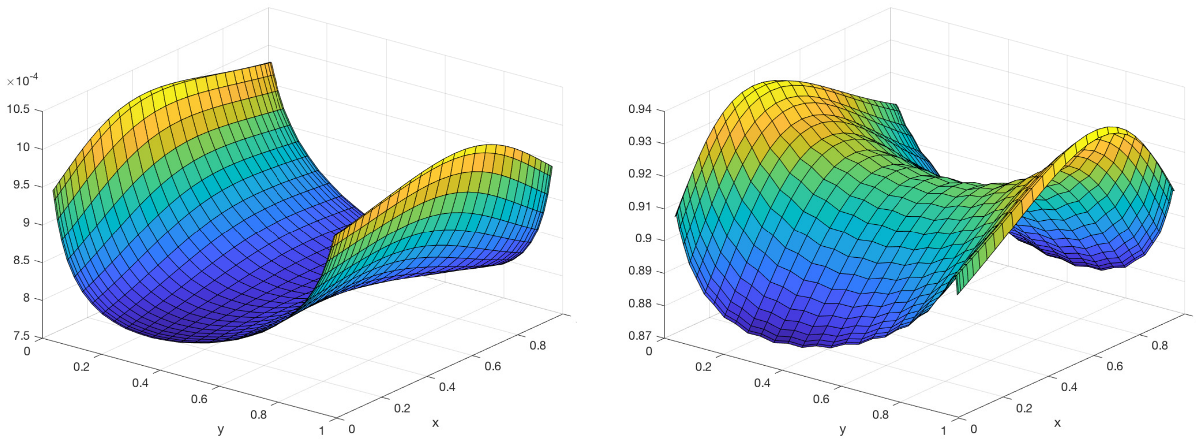


Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.32: Comovement of returns

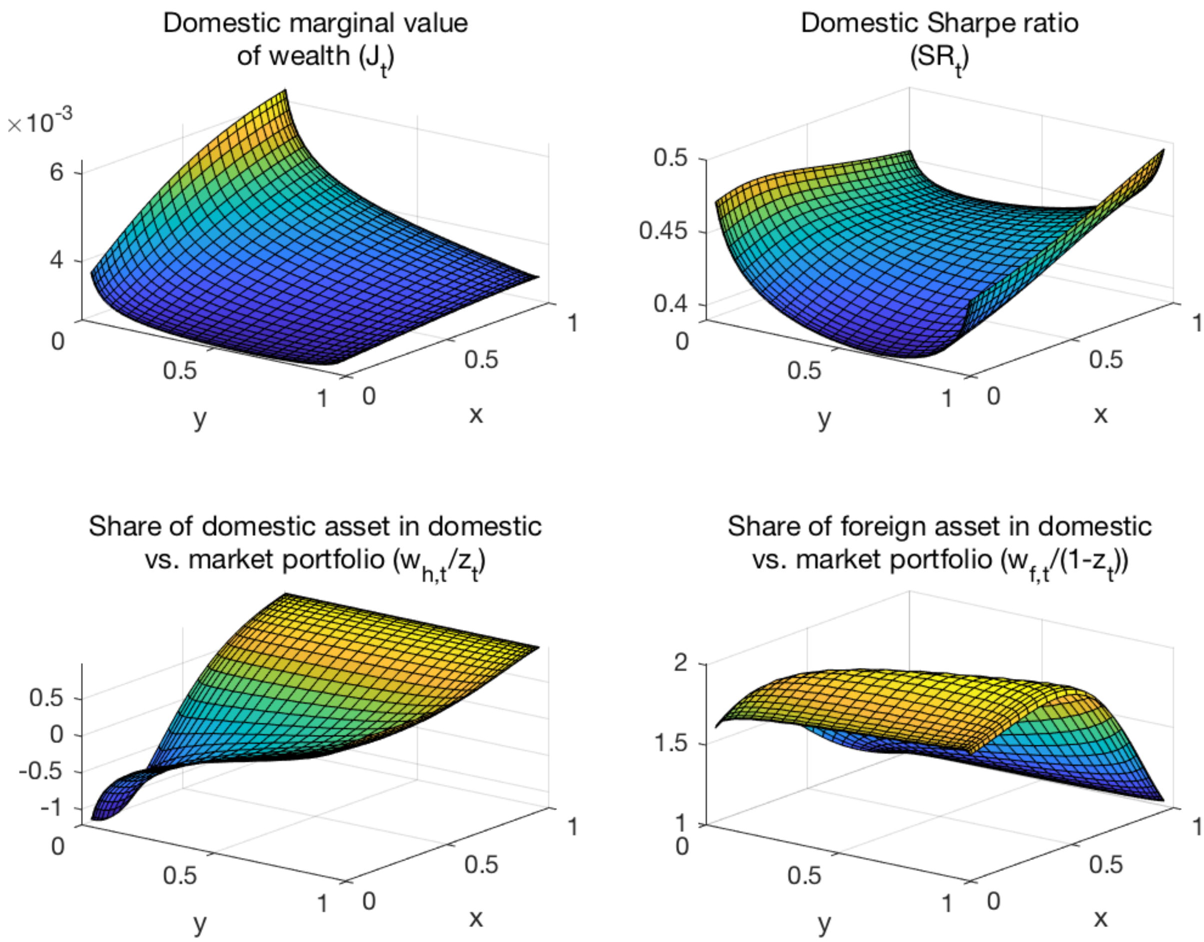
Conditional covariance: $\text{cov}_t(dR_t, dR_t^*)dt^{-1}$

Conditional correlation: $\text{corr}_t(dR_t, dR_t^*)dt^{-1}$



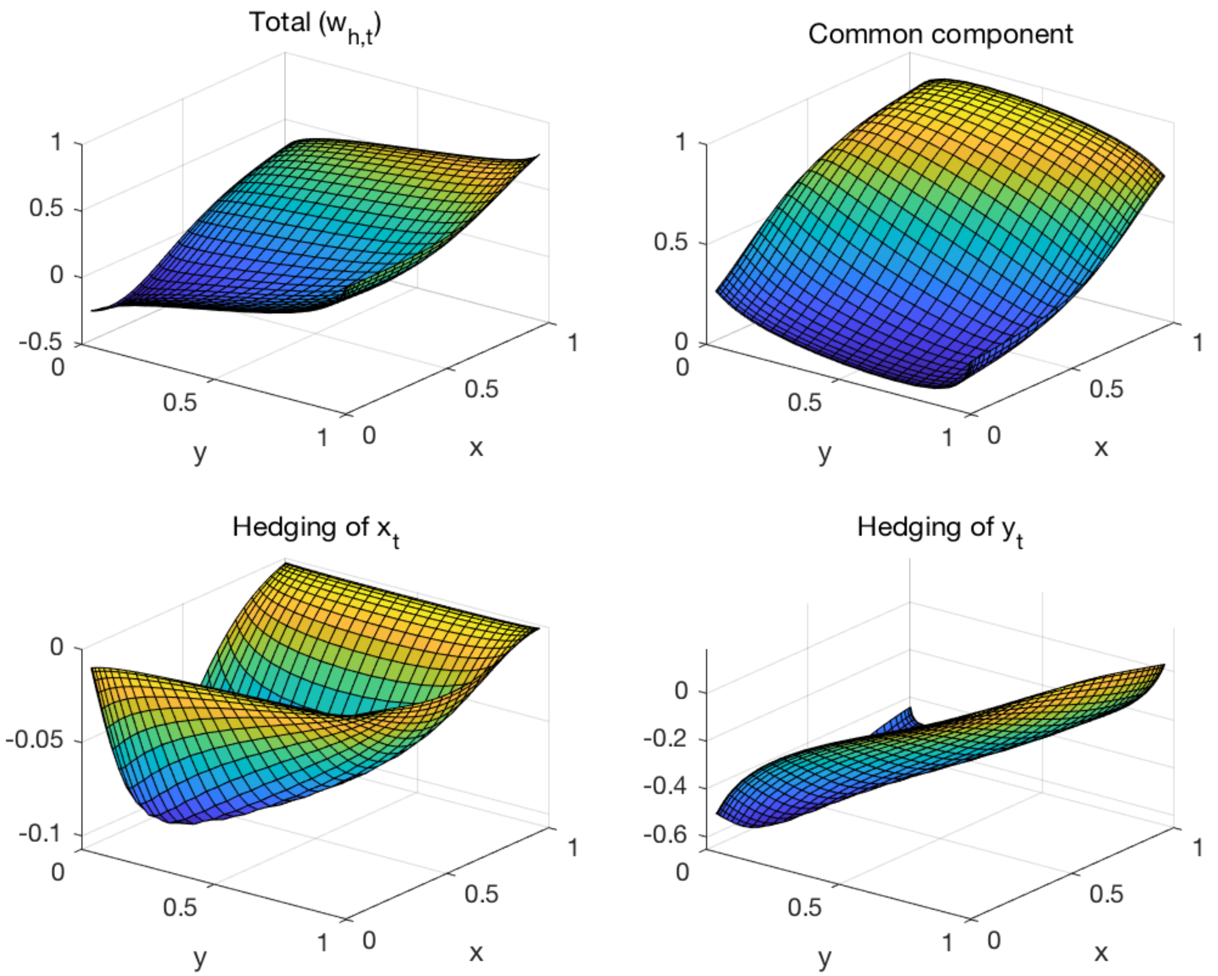
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding two-dimensional representation: Figure 5.

Figure F.33: Direct impact of the wealth share



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding two-dimensional representation: Figure F.14.

Figure F.34: Components of the domestic portfolio (as compared to the market portfolio)



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals. Corresponding two-dimensional representation: Figure 7.